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A USEFUL FORM OF THE STRONG LAW OF LARGE NUMBERS

By

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FOREWORD

In this report the Strong Law of Large Numbers is stated and proved in a form that has been found to be particularly useful in connection with work on the mathematical theory of direct-measurement processes currently in progress in the Statistical Engineering Laboratory of the National Bureau of Standards.

The formulation of the Strong Law here presented, and its proof, were accomplished originally by Professor K. L. Chung while he was a guest worker at the National Bureau of Standards during the summer of 1951. This work was performed under Contract CST-525 between the National Bureau of Standards and the University of North Carolina. Subsequently Professor Chung submitted a somewhat shorter proof that had been communicated to him by Professor William Feller of Princeton University, and it is this more concise proof that is reproduced in this report.

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Theorem: Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with mean 0. Let $\{c_n\}$ be a bounded sequence of real numbers:

$$|c_n| < C$$

Then we have

$$P\left(\lim_{n \to \infty} \frac{c_1 X_1 + \cdots + c_n X_n}{n} = 0\right) = 1$$

In the special case where all $c_n = 1$, this is the well-known Kolmogorov's strong law of large numbers ([1] p. 59).

Proof:* Define

$$X_n' = \begin{cases} X_n & \text{if } |X_n| \leq n \\ 0 & \text{if } |X_n| > n . \end{cases}$$

The sequences $\{X_n\}$ and $\{X_n'\}$ are equivalent, for

$$\sum_{n=1}^{\infty} P(X_n \neq X_n') = \sum_{n=1}^{\infty} P(|X_n| > n) = \sum_{n=1}^{\infty} \int_{|x|>n} dF(x) < \infty .$$

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* This very simple proof was communicated to me by Professor Feller; my original proof was unnecessarily longer.

Next, the series

$$\sum_{n=1}^{\infty} \frac{c_n X_n' - c_n E(X_n')}{n}$$

converges with probability one, by Kolmogorov's criterion (loc. cit.), for

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E(c_n^2 X_n'^2) < C^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{|x| \leq n} x^2 \, dF(x)$$

$$= C^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \int_{k-1<|x|\leq k} x^2 \, dF(x)$$

$$\leq C^2 \sum_{k=1}^{\infty} k \int_{k-1<|x|\leq k} |x| dF(x) \sum_{n=k}^{\infty} \frac{1}{n^2}$$

$$\leq 2C^2 \sum_{k=1}^{\infty} \int_{k-1<|x|\leq k} |x| dF(x) < \infty .$$

It follows as usual that

$$P\left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} C_k(X_k' - E(X_k'))}{n} = 0\right) = 1$$

Hence also

$$P\left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n} C_k(X_k - E(X_k'))}{n} = 0\right) = 1$$

Since $E(X_k) = 0$, clearly $\lim_{k\to\infty} E(X_k') = 0$ and so

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} C_k \, E(X_k')}{n} = 0.$$

Thus the theorem follows.

Corollary 1: Let $\{X_n\}$, $\{\sigma_n\}$, $\{\lambda_n\}$ be altogether independent random variables such that the variables in each set are identically distributed. Suppose that the means all exist and are equal to $E(x)$, $E(\sigma)$, $E(\lambda)$ respectively. Furthermore suppose that the σ_n's are uniformly bounded: $|\sigma_n| \leq N$ with probability one.

Consider the sequence $\{Y_n\}$:

$$\sigma_1 X_1 + \lambda_1, \ldots, \sigma_1 X_{n_1} + \lambda_1;$$

$$\sigma_2 X_{n_1+1} + \lambda_2, \ldots, \sigma_2 X_{n_2} + \lambda_2;$$

$$\cdots\cdots\cdots$$

$$\sigma_k X_{n_{k-1}+1} + \lambda_k, \ldots, \sigma_k X_{n_k} + \lambda_k;$$

$$\cdots\cdots\cdots$$

where $n_1 < n_2 < \ldots < n_k \ldots$, and $n_k - n_{k-1} \leq M$. Then the strong law of large numbers holds for $\{Y_n\}$ as follows:

$$P\left(\lim_{n \to \infty} \frac{Y_1 + \ldots + Y_n}{n} = E(\sigma)E(X) + E(\lambda) \right) = 1.$$

Corollary 2: As in Corollary 1, consider the sequence $\{Z_n\}$:

$$a_1 Y_1 + \ldots + a_m Y_m, \ldots, a_1 Y_{n_1-m+1} + \ldots + a_m Y_{n_1};$$

$$a_1 Y_{n_1+1} + \ldots a_m Y_{n_1+m}, \ldots a_1 Y_{n_2-m+1} + \ldots + a_m Y_{n_2};$$

$$\cdots\cdots\cdots$$

$$a_1 Y_{n_{k-1}+1} + \ldots + a_m Y_{n_{k-1}+m}, \ldots, a_1 Y_{n_{k-1}-m+1} + \ldots + a_m Y_{n_k};$$

where the a_i's are real constants, $|a_i| \leq A$. Then the strong law of large numbers holds for $\{Z_n\}$ as follows:

$$P\left(\lim_{n \to \infty} \frac{Z_1 + \ldots + Z_n}{n} = (a_1 + \ldots + a_m)\{E(\sigma)E(\lambda) + E(\lambda)\}\right) = 1.$$

[1] A. N. Kolmogorov. Grundbegriffe der Wahrscheinlichkeits-rechnung, Chelsea Publishing Co., New York.