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NBS-INA — The Institute for Numerical Analysis — UCLA 1947–1954

Magnus R. Hestenes and John Todd

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Magnus R. Hestenes
Department of Mathematics
University of California
at Los Angeles
Los Angeles, CA 90078

John Todd
Department of Mathematics
California Institute of Technology
Pasadena, CA 91109

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ABSTRACT

This is a history of the Institute for Numerical Analysis (INA) with special emphasis on its research program during the period 1947 to 1956. The Institute for Numerical Analysis was located on the campus of the University of California, Los Angeles. It was a section of the National Applied Mathematics Laboratories, which formed the Applied Mathematics Division of the National Bureau of Standards (now the National Institute of Standards and Technology), under the U.S. Department of Commerce.

This history of the program at INA is concerned primarily with the development of mathematics pertinent to solving numerical computations. This development could happen only if some mathematicians were proficient in handling the electronic digital computers. To insure that there would be some, INA was constituted. It was well funded, and could attract first class mathematicians to take a year off for research at INA. There they were in the midst of people solving problems of considerable difficulty using the digital computers. They were thus enticed into using them. When this happened, many important developments emerged. This history is centered around these people and asks "Who were there?", "What were their interests?", "What did they do?".

Key words: differential equations; digital computers; history; linear programming; matrix; numerical analysis; UCLA

FOREWORD

In the 1930's A. M. Turing conceived the idea of a computer. The electronic technology needed to construct a working model did not then exist. However, during World War II great strides were made in electronic technology. Just after World War II there appeared the ENIAC. It was an electronic computer but not of the Turing type. However, the Turing computer was obviously just around the corner, and would run rings around the ENIAC. It was anticipated that the Turing computer would give much assistance in various mathematical studies.

This could happen only if some mathematicians were proficient in handling the Turing computer. To assure that there would be some, the Institute for Numerical Analysis (INA) was constituted. It was well funded, and could attract first class mathematicians to take a year off for research at INA. There they were in the midst of people solving problems of considerable difficulty by the use of the Turing computers. They were enticed into learning to use them. When this happened, many important developments emerged, as recounted in the history that follows.

Summer 1985

J. Barkley Rosser
Professor Emeritus
Center for Mathematical Sciences
University of Wisconsin, Madison

PREFACE

Over 30 years have passed since the events described in this History took place. We have recounted them to the best of our recollection and apologize for our sins of omission and commission. We have given references to our sources and the material which we have used will be deposited in appropriate places. The Quarterly Progress Reports and internal memoranda of the Institute for Numerical Analysis (INA) were particularly useful. Attention is invited to the Proceedings of the 1987 Princeton Conference [131] in which the articles by G. B. Dantzig and ourselves are particularly relevant.

Our point of view is probably biased towards the West and may not represent the official Washington attitude. Further, in part because the original sponsor of the writing is the Mathematical Association of America, we have stressed the research and educational aspects of the program, rather than the machine development and computational service aspects, although they had undeniable importance. Finally, our selection of topics for more detailed discussion is undoubtedly different from that of others who might have undertaken this work—differential equations, Monte Carlo, combinatorics, number theory could well have been emphasized more.

We are grateful to many of our former colleagues who shared their reminiscences with us, and we are grateful for the opportunity to communicate to a wider public some account of what was an exciting time for both of us.

The machine development aspect of INA has been well documented. A short history of this development is reproduced in Appendix C. This history was written by H. D. Huskey, who was responsible for the building of the SWAC, the National Bureau of Standards Western Automatic Computer. There is, however, no adequate documentation of the research aspect of INA. Accordingly, we give here a history of the program at INA concerned primarily with the development of mathematics pertinent to solving problems involving numerical computations. We include abbreviated remarks about other aspects of the program at INA. Our story is centered around people. We ask "Who were there?", "What were their interests?", "What did they do?". We give only selected answers to the questions with the hope that they are sufficient to give a bird's-eye view of the program at INA.

In a sense the researchers at INA pioneered in modern numerical analysis. "Graduates" of INA are scattered all over the United States and elsewhere. They have done much to promote the development of numerical mathematics pertinent for use of computing machines for scientific and educational purposes.

We wish to single out for special mention three persons who played a dominant role in the formation and maintenance of INA. They are E. U. Condon, John H. Curtiss, and Mina Rees. Condon was the Director of the National Bureau of Standards (NBS) and was an enthusiastic supporter of the project. He gave the "go ahead" for Curtiss to set up INA as part of the Applied Mathematics Program of the NBS. Curtiss was the guiding force of the INA project and was instrumental in interesting prominent mathematicians to participate in the program. Mina Rees was the Head of the Mathematics Branch of the Office of Naval Research (ONR). She was one of a group which saw need for the development of "machine" mathematics and encouraged Curtiss to proceed with his program. She was instrumental in arranging substantial financial support by ONR for the INA project. In fact, ONR funded the major portion of the mathematical research program of INA. The machine development program was largely supported by the Air Force. Contributions to this program were also made by the Army. These governmental agencies played an important role in the formation of INA.

There are several mathematicians to whom we are especially grateful for their help in the preparation of this volume. First, Olga Taussky-Todd, who was one of the original crew in 1947–48 and was a consultant to the NBS Applied Mathematics Division (AMD) from 1949 to 1957. Her

wide knowledge of mathematics and mathematicians played an important part in the development of NBS-INA, as will become apparent in our history. In addition, her outstanding memory and her collection of INA photographs have been invaluable during the preparation of this history.

We are also indebted to Marion I. Walter, who was at INA in 1952–53 for her extensive collection of INA photographs.

We also acknowledge the permissions to reproduce published photographs which have been given by various organizations and individuals.

We appreciate comments on drafts of our manuscripts by E. W. Cannon, C. R. De Prima, and J. H. Wilkinson.

Finally we appreciate greatly the work of Churchill Eisenhart and Joan Rosenblatt who carefully read our manuscript and tried to make us historians of science and of Shirley Bremer who collated and checked all the changes and corrections and acted as our representative with the NBS production staff who, under the direction of Don Baker, Rebecca Pardee, and Ernestine Gladden, did a splendid job.

M. R. Hestenes
UCLA
Los Angeles, CA 90024

John Todd
Caltech
Pasadena, CA 91125

CONTENTS

CHAPTER I

THE BEGINNINGS

This is a story of the Institute for Numerical Analysis (INA) with special emphasis on its research program during the period 1947 to 1954. The Institute for Numerical Analysis was located on the campus of the University of California, Los Angeles (UCLA). It was a section of the National Applied Mathematics Laboratories (NAML), which formed the Applied Mathematics Division (AMD) of the National Bureau of Standards (NBS), which was a part of the United States Department of Commerce.

Harry S. Truman was President from 1945 to 1952 and Dwight D. Eisenhower was President from 1952 to 1960. E. U. Condon was the Director of NBS from 1944 to 1951 and A. V. Astin was the Director from 1951 to 1969. Although the Directorship of NBS was a Presidential appointment, it was regarded as non-partisan.

The NBS was established in 1901 and the basic legislation was greatly amended in 1950. Included in the 1950 amendment was authorization to "operate a laboratory of applied mathematics." This was permissive, not mandatory. In 1951 a Semi-Centennial celebration was held. At that time a history of NBS was written by Rexmond C. Cochrane [12]. As part of this celebration a Symposium was held at INA on "Simultaneous Linear Equations and the Determination of Eigenvalues" [see Appendix D]. This Symposium was the first of a long series of Symposia on this and related subjects. The contributions of these Symposia to the development of modern numerical analysis were significant.

While it is tempting to trace the origins of INA back to Archimedes, or Pythagoras, or to Leonard da Vinci, or to Peter the Great, or to other scientists who worked for their governments, it is more realistic to begin with World War I. Governments realized that some scientists could contribute more in their laboratories than in the field. Accordingly, scientists were recruited to work on special programs of scientific and mathematical nature. We restrict our remarks to mathematical aspects of these programs. In the United States, for example, F. R. Moulton, G. A. Bliss, O. Veblen, and others were stationed at the Aberdeen Proving Grounds to study problems in ballistics. At this time Bliss developed a method of differential corrections for predicting the trajectories of shells fired from a long range cannon. This method proved to be very effective. When World War II was imminent, Bliss resurrected this method in a graduate course on exterior ballistics and wrote a book on this subject. The method of differential corrections is a generalized Newton's method applied to differential equations. It is a quasilinearization method of the variety used later by R. Bellman.

R. Courant made scientific contributions to the German war effort, even though he spent considerable time on the front lines. His experiences are vividly described by Constance Reid [79].

In England, J. E. Littlewood was called upon to apply his talents to the study of ballistics. He was stationed in London, because "The higher brass at Woolwich recognized that Littlewood should not be subjected to routine chores and petty restrictions. He could 'live out' and make his home with friends in London. If, in uniform, he carried and used an umbrella, it would not be seen." His *Adventures in Ballistics* can be found in his Collected Papers [57]. Other mathematicians, who have made significant contributions to Numerical Analysis, were not so fortunate. A. C. Aitken was wounded and invalided. An account of Aitken's experiences can be found in his autobiography [3] and his obituary [3]. L. J. Comrie was also wounded as noted in his obituary [128].

Between World War I and World War II, there was little activity in the development of National Mathematical Laboratories. However, during the Depression, the Works Progress Administration set up a Mathematical Tables Project in New York, under the scientific direction of the National Bureau of Standards. It was staffed largely by unemployed high school mathematicians (about 100), supervised by a small group of mathematicians such as A. N. Lowan, Gertrude Blanch, Ida Rhodes, M. Abramowitz, H. E. Salzer, and Cornelius Lanczos. The story of this group was given in articles by Lowan [60] and by Blanch and Rhodes [6].

In Italy, M. Picone organized the Istituto Nazionale per le Applicazioni del Calcolo (INAC). It was founded in Naples in 1927 and established in Rome in 1932. Picone [72] characterizes INAC as a "living table" of functions. The general problem of numerical analysis is to compute $f(x)$ given x, granted reasonably generous interpretations of f and x: for instance, x might be a matrix and f the set of its characteristic values and vectors. In about 1952, Picone [73] discussing the accomplishments of INAC wrote that "It is the place where the marriage between functional topology and numerical calculation has taken place." At this time there were close relations between INAC and NBS-AMD. Picone and G. Fichera made brief visits and E. Aparo and D. Dainelli longer visits to NBS-AMD. Fichera and Aparo made extended visits to INA. NBS-AMD returned these visits and contributed to conferences organized by INAC.

The War Years 1939–1945

World War II began on September 3, 1939 for the United Kingdom and on December 7, 1941 for the United States. VE Day was May 7, 1945 and VJ Day was August 14, 1945.

Initially, it was not clear where scientists could be most usefully deployed. Some were more adaptable and some were more imaginative and aggressive and found suitable slots for themselves. In the United Kingdom in the summer of 1940, a census of scientists and engineers was made under the direction of C. P. Snow. Some of the civil servants involved seemed more interested in the novelty of compulsory registration announced on radio, in some way going back to the town crier, than in the actual classification. Many of the academics, who participated in this census, were able to find for themselves employment suited to their experience.

During these years in the United Kingdom and the United States, existing groups with mathematical strengths were built up and new groups developed around energetic mathematicians. For example, in the United Kingdom, we note the very specialized cryptographic work at Bletchley, which had a great impact later. For this see *Annals of the History of Computing 1-*, 1979-. We note also the Differential Analyser Group at Manchester, whose activities have been described by D. R. Hartree [26]. Another organization was the so-called Admiralty Computing Service (ACS) which was developed, beginning in 1942, by D. H. Sadler and John Todd. A description of its activities is given in their article [87]. Important contributions were also made by a commercial organization, namely, Scientific Computer Service Ltd., which was organized by L. J. Comrie and with which J. C. P. Miller and H. O. Hartley were associated [128]. A report on British Intelligence [33] credits Comrie for locating an important enemy radio installation in France in 1940.

The war-time activities of mathematicians in the United States are well documented in articles by M. S. Rees [77] and J. B. Rosser [84].

We note that the ENIAC was developed at the Ballistics Research Laboratories at the Aberdeen Proving Grounds under the supervision of the University of Pennsylvania. The ENIAC was the first large-scale electronic automatic computing machine built in the United States.

Among the many scientific organizations engaged in war-related activities was an Applied Mathematics Panel with Warren Weaver as Chief. It was set up by the National Defense Research Committee. This panel had several groups working at various universities and other institutions. One of us (Hestenes) was attached to the Columbia University Group. Another of the groups was the Mathematical Tables Project, under Arnold Lowan, which later became the basis of the NBS Computation Laboratory.

Among the mathematical leaders in Germany were Alwin Walther, Robert Sauer, and Lothar Collatz. There was developed a National Research Institute at Oberwolfach under the direction of Wilhelm Süss. An account of this Institute can be found in the articles by Irmgard Süss and by John Todd in a book edited by E. F. Beckenbach and W. Walter [4]. Extensive accounts of mathematical activities in Germany from 1939 to 1946 were issued as Field Information Agencies Technical Reports [103,123] edited by W. Süss and A. Walther respectively.

The Post-War Years

During the time of reconstruction, established organizations, such as the Picone Institute and the Oberwolfach Institute, developed within the then existing limitations.

Before discussing in some detail the beginnings of the national organizations in the United Kingdom and in the United States, of which we have some personal knowledge, we briefly mention two European centers, a program in the U.S.S.R., and an international development. We shall see that both the British and American organizations owed their existence in large part to the respective naval establishments. For an account of the American contributions we refer to two papers by Mina S. Rees [75,76].

In 1948 a Mathematisch Centrum was set up in Amsterdam to deal with problems in pure and applied mathematics, statistics, and the construction and use of computers. Among those active in this center were D. van Dantzig, A. van Wijngaarden, and J. G. van der Corput. This organization has made and continues to make notable contributions in all its areas of interest, e.g., in the study of North Sea floods.

In 1947 the Swedish government set up an organization to develop and use computers. This, too, has had considerable influence, as indicated, for example, by the relatively high number of Swedish mathematicians specializing in numerical mathematics, e.g., C. E. Fröberg, G. G. Dahlquist, H. O. Kreiss, and V. Thomée.

The immediate post-war program in the U.S.S.R. is described by S. Vavilov in "Our Five-Year Plan for Science," *Soviet News*, 26 September, 1946. We quote:

> "Mathematics, which is of vital importance to natural science, to technique, and to such social sciences as economics, is directly linked up with problems of philosophy and logic. Much of our five-year plan for mathematics is directed towards assisting other sciences. For example, we are stressing questions of the theory of probability—particularly those bearing on the interpretation of observations and research on partial differential equations, particularly those associated with what may be termed "machine mathematics," that is, the solution of mathematical problems with the help of calculating devices.
>
> Calculating machines have been known for centuries. But never has "machine mathematics" reached such scope as at the present time. New calculators devised on electrical principles make it possible to solve extremely complex mathematical problems connected with technique and the various branches of natural science. So important do we think this side of mathematics is, that we are proposing in the immediate future to devote to it a special institute.
>
> However, machines can never displace mathematical thought. A characteristic of mathematical thought is its boldness, its imaginative power. Such creative mathematics, which at times find no immediate application in technical science, has always found fertile ground in our Academy; and its development must continue on a broad scale. Such subjects as non-euclidean geometry, the tensor calculus, and the theory of groups, which seem to be abstract studies absolutely cut off from life and reality, nevertheless suddenly assume a decisive significance at definite stages of scientific development.
>
> This explains the inclusion in the Academy's plan of the problems of the theory of numbers, abstract algebra, topology, and mathematical logic."

Much later an international activity was conceived by UNESCO. During the years of planning and competition for a permanent International Computing Center (ICC) an ICC Preparatory Committee organized several symposia in the late 1950's (Proceedings of some were published by Birkhäuser Verlag, Basel). Among the leaders in the negotiations were Stig Comét, R. de Possel, A. Ghizzetti, and C. Berge. A Provisional ICC (PICC) commenced operations in Rome in January 1958. The ICC was finally set up in November 1961. Both the PICC and ICC published *Bulletins*; 16 issues 1958–1962 by PICC and 6 volumes 1962–1967 by ICC.

In the United States, a private research organization called the Rand Corporation was formed. Its mathematics branch had close ties with INA, particularly in the study of Theory of Games, Linear Programming, Computers, and related fields. One of us (Hestenes) was a consultant to E. Paxson at Rand from 1947–1950. In his study of time-optimal flights of a fighter plane, he was led to a formulation of a general Optimal Control Problem and gave basic conditions for its solution. In a similar study of encounters of fighter planes, the Russian mathematician L. S. Pontryagin was led to an Optimal Control problem of the same type. His formulation of the basic conditions that a solution must satisfy is now called "Pontryagin's Maximum Principle." At Rand, R. Bellman, in the study of a similar problem, was led to the theory of Dynamical Programming. R. Isaacs at Rand developed a theory of Differential Games, as did Pontryagin. Studies of these topics were also pursued at Rand by L. Berkovitz. The group at Rand was computer oriented and sought computer solutions to their problems when it was feasible to do so. They also made significant contributions in the field of Mathematical Programming.

We turn now to a general discussion of the founding of the national mathematics laboratories in the United States and in the United Kingdom. These laboratories were constituted as units of the National Bureau of Standards (NBS) in the United States and of the National Physical Laboratory (NPL) in the United Kingdom. Their location and their immediate successes stemmed from the fact that the then heads of the NBS, E. U. Condon, and the NPL, C. G. Darwin, were substantial mathematicians, though technically physicists. These two mathematical laboratories became in a sense "sister" institutions in that they kept in close contact and staff members made frequent visits, one to the other.

C. G. Darwin's interest in special functions is evidenced by his paper on "Weber's Function" [17] and by the tables of J. C. P. Miller [62]. John Todd recalls Darwin telling him that, on his way to the United States (to head the British Commonwealth Scientific Office) during wartime on one of the Queens, he told the captain that he had worked out, with primitive observations and primitive computing equipment, the position of the ship, for which he was promptly reprimanded. Details of Darwin's life and work can be found in the obituary by Sir George Thompson [113].

In 1945 those concerned with the Admiralty Computing Service arranged, through appropriate channels, for an invitation for the Department of Scientific and Industrial Research, of which Sir Edward Appleton was the head, to consider the establishment of a National Mathematics Laboratory (NML), in parallel to the National Physical Laboratory (NPL). To support this suggestion a memorandum was prepared by A. Erdélyi, D. H. Sadler, and John Todd. It was entitled "Memorandum on the centralization of computation in a National Mathematical Laboratory." See also [130]. Two related articles were also prepared, one by D. H. Sadler and John Todd and one by A. Erdélyi and John Todd. After various negotiations, the NML was realized as a Division of NPL with J. R. Womersley as its first Chief (1945–1954). Several members of the computing staff of the ACS trained by D. H. Sadler, including L. Fox and F. W. J. Olver, joined the new organization. Among the other members of the staff of this Division were A. M. Turing, J. H. Wilkinson, and E. T. Goodwin (Chief from 1954–1972).

Return now to the story of E. U. Condon and the NBS. As remarked above Condon was a substantial mathematician as well as a distinguished physicist. As a person, he was compassionate, outspoken, but controversial, especially in the McCarthy era. All significant scientists have international contacts and this led to the characterization of Condon as the weakest link in the chain of security by Congressman J. Parnell Thomas (who was later jailed for taking kickbacks from his staff). His responses in hearings were legendary. We recall how, on a visit to a major university, he encountered an unemployed scientist, living in the Common-Room. On his return he asked us to try to find an opening for him—which we did and he made notable contributions to the INA.

For details of Condon's activities we refer to the sympathetic obituaries by P. M. Morse [64] and by Churchill Eisenhart [18]. Eisenhart has been with the Applied Mathematics Division (AMD) in various capacities since its beginning. Morse of MIT was an enthusiastic supporter of the work of the AMD from its earliest days and, for example, was chairman of the committee which organized the famous *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun [1].

We finally come to the story of the establishment of the National Applied Mathematical Laboratories (NAML) and the choice of UCLA as the location for the Institute for Numerical Analysis (INA).

At the end of World War II farsighted leaders in the U.S. Government deemed it to be important that scientific research be promoted in order to meet the future needs of society. Accordingly, various agencies were set up in order to promote research. In particular the Navy established the Office of Naval Research (ONR). In 1946 the Mathematics Branch of ONR was formed and Mina Rees was selected to be its head. At about the same time John Curtiss was chosen to be the Assistant to the Director, E. U. Condon, of the National Bureau of Standards (NBS). Mina Rees and John Curtiss were largely responsible for the formation of INA. In doing so they frequently sought advice from C. B. Tompkins, a mathematician whose war experience convinced him that high-speed digital computers were on the horizon and needed to be developed. In 1953 Tompkins became the Director of INA.

Early in 1946 ONR requested the assistance of NBS in the establishment of a national mathematical computation center. Cooperative studies of this proposal revealed the desirability of a more fundamental approach. Specifications for a more general facility evolved after lengthy consultations with scientific representatives of industrial groups, education institutions, and other governmental agencies. Early in 1947 NBS issued a prospectus for the formation of National Applied Mathematical Laboratories (NAML) to be guided by a committee of representatives of interested outside groups, to be called the Applied Mathematics *Executive* Council. (In 1949 this was rechartered as the Applied Mathematics *Advisory* Council.) In this prospectus it was proposed that these laboratories consist of four major units. The first of these was a section devoted primarily to research and training, to be located in California and to be called the Institute for Numerical Analysis. We are not clear about the reasons for choosing a location in California rather than a more central one. Perhaps Condon was remembering his student days at Berkeley. Perhaps the climate was a main attraction. A certain distance from Washington was certainly desirable, for some mathematicians are uncomfortable with strict dress code and regular hours. The second laboratory was a large Computing Laboratory to be located in Washington, DC, or in New York. The third unit was a Statistical Engineering Laboratory, and the fourth was a Machine Development Laboratory devoted to the development of automatic digital computing machinery. The last two units were to be located at the NBS in Washington, DC. When these laboratories were formed in 1947, the second was also located at NBS and the first was scheduled to be located at UCLA.

Late in the fall of 1946 John Curtiss came West on a mission for E. U. Condon to seek a home for the proposed INA. He first visited Berkeley and Stanford and then came south to UCLA. At the request of Provost Clarence A. Dykstra, Samuel Herrick of the UCLA Astronomy Department arranged for Curtiss' UCLA presentation of the proposal to set up an Institute for Numerical Analysis on the west coast and to set up sister applied mathematical and numerical laboratories at NBS. Edwin F. Beckenbach, in particular, was excited about the possibility of having such an Institute located at UCLA with a flow of outstanding mathematicians as visitors. He took the initiative for writing a proposal to NBS for the location of INA on the UCLA campus. In doing so he was ably assisted by Ivan Sokolnikoff. John Curtiss also visited Lee Du Bridge, the President of Caltech, who said he had no space available for an INA project but would be glad to support the UCLA proposal. After the proposal by UCLA had been accepted by NBS, Beckenbach became an early participant of Project INA and Sokolnikoff became a local advisor.

According to the proposal to NBS, the primary function of the INA would be to conduct research and training in the types of mathematics pertinent to the efficient exploitation and further development of high-speed automatic digital computing machinery. A secondary function would be to provide expert computing service for local groups with immediate computational problems, and to assist, as necessary, in formulations and analysis of difficult problems in applied mathematics. To carry out these functions INA would be supplied with one general-purpose automatic electronic digital computing machine of large capacity together with desk calculators and punch-card equipment as needed. More precisely the functions of INA would be the following:

(a) Plan and conduct a program of research in pure and applied mathematics directed primarily at developing methods of analysis which will permit the most efficient and general use of high-speed automatic digital computing machinery.

(b) Conduct training programs for personnel of industry, Government agencies, and educational institutions, in the theory and disciplines needed for the full exploitation of high-speed automatic digital computing equipment.

(c) Study and formulate requirements for the intelligence and internal organization which high-speed automatic computing machinery should have; develop overall performance specifications for such machinery.

(d) Serve as a center at which competent scholars can explore the usefulness of high-speed automatic digital computing machinery in their own fields of interest.

(e) Formulate requirements for further mathematical tables and other aids to computation; review the overall program of the National Applied Mathematical Laboratories with regard to the production of such objects and advise the Administration and Executive Council accordingly.

(f) Review, analyze, and, as necessary, assist in the mathematical formulation of problems in applied mathematics of the more complex and novel type arising in outside laboratories.

(g) Provide a computing service containing both standard and high-speed automatic equipment (when available) for local industries, educational institutions, and Government agencies.

(h) Assist and conduct liaison with related programs in local educational institutions.

(i) Maintain a consulting service on special problems in applied mathematics.

(j) Prepare reports of the research described above; also prepare training manuals, bibliographies, and indices.

The development of the AMD and, in particular, the INA is described in the (initially) quarterly reports; "Projects and Publications of the National Applied Mathematics Laboratories," which were given a quite wide distribution. Projects with managers, objectives, backgrounds, priorities, and magnitude were set up, continued, and terminated. Progress on these was reported quarterly. The preparation of these reports was quite arduous, for mathematicians were not yet conditioned to the preparation of grant proposals (the NSF was not set up until 1950).

The name NAML was dropped in 1954 in deference to the views of other organizations who aspired to a national title.

Related papers on the history of the Institute for Numerical Analysis have appeared in [134–136].

CHAPTER II

THE PERIOD SUMMER 1947 THROUGH SUMMER 1948

The National Applied Mathematics Laboratories were formally organized as Division 11 of the National Bureau of Standards on July 1, 1947. The plan of organization followed closely the one set forth in the "Prospectus" and approved by various officially interested agencies. John H. Curtiss was made their Chief. It was decided to locate the Institute for Numerical Analysis at UCLA. However, INA was temporarily housed at NBS in Washington, DC, awaiting the completion of adequate facilities at UCLA. The first appointees to INA were Olga Taussky-Todd, John Todd, and Albert Cahn. The first two projects sponsored by INA were entitled "Characteristic roots of matrices" and "Applications of automatic digital computing machines in algebra and number theory." The Todds were their project managers. These projects were the first of many to which the researchers at INA made significant contributions. In a sense they signalled the beginning of a new endeavor which later became known as Computer Science. Cahn's duties were largely administrative. Otto Szász joined the group in February 1948. His specialty was the theory and applications of infinite series.

John Todd's first contact with NBS-INA was a letter from Curtiss, dated April 16, 1947, describing the plans and inviting him to spend the next academic year at the "Institute" performing research in the field of his choice—the sole condition being that the research should have some general bearing on the functions of the "Institute" as set forth in the Prospectus. Olga Todd was appointed as soon as she arrived in Washington, in the fall of 1947.

Szász and the Todds spent the first few months of 1948 at the Institute for Advanced Study in Princeton, at the invitation of John von Neumann. During that period Olga Todd, in discussion of questions raised by S. D. Chowla, J. Nielsen, and I. Reiner, saw the relevance of a theorem of Latimer and MacDuffee [52] on integral matrices. Chowla encouraged her to find a simple proof of this. This was accomplished at INA in the summer of 1948, using the characteristic vectors of the matrices. This is now accepted as the definitive proof. She mentioned this to Hans A. Rademacher [74], who spent the summer at INA. He immediately found a use for it in his work on Dedekind sums. Rademacher gave a series of expository lectures on elliptic and modular functions. In addition he worked on the accumulation of round-off errors in numerical computations.

The new building for the Institute for Numerical Analysis at UCLA was occupied in April 1948. The Todds, Otto Szász, and Gertrude Blanch were the initial members. Blanch was an expert on Mathieu Functions. She organized the computational unit. Almost immediately computational services were performed for the Departments of Astronomy and Engineering. In particular, Rocket Navigational Tables were constructed at the request of Samuel Herrick of the Astronomy Department. The period April–June 1948 was an organizational period in which the scientific and computational staffs were acquired. Roselyn A. Siegel became the first junior member of the computational unit.

The table-making tradition of the NBS was continued at INA. Gertrude Blanch was largely responsible for ensuring that the tables issued were of the quality (typographic and accurate) which the scientific public had been led to expect. The following tables in the NBS Applied Mathematics Series (AMS) are among those which were almost wholly the work of INA. (Some of the copy was prepared on a card controlled typewriter at the U.S. Naval Observatory.)

AMS 11 (1951) Tables of Rational Arctangents, by John Todd. Reprinted as a paperback in 1965.

AMS 20 (1953) Tables for Rocket and Comet Orbits, by Samuel Herrick.

AMS 50 (1959) Tables for the Bivariate Normal Distribution Function and Related Functions.

The tables of contents of these publications are given in Appendix D.

Initially all computations were carried out on standard desk calculators. The Institute quickly acquired an IBM Card Programmed Calculator, dubbed a CPC. This calculator was programmed on a small plug board. Input was accomplished by reading punched cards. It had a typewriter output. If one wished, one could print out the result of each step in the computation. Doing so was an education to the user because it clearly demonstrated the effect of roundoff errors. It showed the need for suitable scaling of numbers and the advantages of using floating point arithmetic. Originally it was planned to purchase a high-speed automatic digital computer for use at INA. This was because NBS was not permitted to compete with industry by building its own computer. However, noncompetitive experimental models could be constructed. Because no automatic computers would be available in the near future, it was decided to build an innovative computer at INA based on the use of a Williams tube memory. Harry D. Huskey was in charge of its development. It became known as the SWAC, the *S*tandards *W*estern *A*utomatic *C*omputer. Construction of the SWAC began in January 1949. It was dedicated in August 1950 and decommissioned in 1967. (For details see Appendix C.)

Douglas R. Hartree, a Mathematical Physicist from Cambridge, England, was invited to be the first Director of INA. Unfortunately, he could stay only a short time. The invitation of Hartree was a natural one because he was a pioneer in machine computations. In the 1920's he devised a method of Self-Consistent Fields to solve problems which emerged with the advent of wave mechanics. This is basically an iterative method of successive approximations. The numerical labor was enormous. Upon learning about the Differential Analyzer (D.A.) being developed by Vannevar Bush at MIT, he realized that the D.A. could be exploited for his problem. Accordingly, he pioneered in the development of D.A.'s in England for scientific computation. Hartree and his collaborators made effective use of the D.A. to solve a great variety of problems before and during World War II. During World War II, the ENIAC was developed in the United States at the University of Pennsylvania. Initially, its main function was the determination of trajectories for the Aberdeen Proving Grounds. Hartree visited the University of Pennsylvania and solved two-point boundary value problems on the ENIAC. He also visited the Aberdeen Proving Grounds and lectured on computing instruments like the D.A. and the ENIAC. Upon returning to England, he kept in close touch with the development, by M. V. Wilkes, of automatic electronic computing machines (the EDSAC Project) in England.

It was Hartree's experience that, by and large, the mathematical community was inexperienced in numerical analysis and showed very little interest in the subject. Some even belittled it. In 1948 in an INA Symposium (see AMS 15 in Appendix D) Hartree said, "One of the unsolved problems of Numerical Analysis is how to overcome the attitude of the Mathematical Fraternity on this subject." Of course, one of the purposes of the NBS Laboratories and, in particular, INA was to solve this problem. That this problem has been solved is in large measure due to the influence of John Curtiss and the support given him by NBS and by various governmental agencies under the leadership of Mina Rees of ONR.

During Summer 1948 the research program of INA was carried out intensely under the direction of D. R. Hartree with the assistance of a number of distinguished mathematicians on temporary appointments. Supplementing the research program, a set of "Symposia on Modern Calculating Machinery and Numerical Methods" was held on July 29–31, 1948. Sponsorship of the Symposia was undertaken jointly by UCLA and INA. The agenda involved 32 addresses and various other round-table discussions. The topics ranged from reports on the status of machine construction projects to the statements of mathematical problems awaiting high-speed automatic computing machinery for solution. The opening address was given by John von Neumann. Douglas R. Hartree spoke on "Some unsolved problems in numerical analysis." Solomon Lefschetz talked about "Numerical calculations in nonlinear mechanics." Bernard Friedman presented a paper on "Wave propagation in hydrodynamics and electrodynamics." George B. Dantzig discussed "Linear programming." This symposium served as dedicatory exercises for the Institute and the proceedings were published as AMS 15 (see Appendix C).

SUMMER 1948

The roster of the senior research personnel for Summer 1948 was the following:

Douglas R. Hartree	(Cambridge, England; Acting Director)
Edwin F. Beckenbach	(UCLA representative)
Gertrude Blanch	(Head of Computational Unit)
Robert H. Cameron	(Minnesota)
George E. Forsythe	(INA)
Robert E. Greenwood	(Texas)
Samuel Herrick	(UCLA Astronomer)
William E. Milne	(Oregon)
Hans A. Rademacher	(Pennsylvania)
Wladimir Seidel	(Wayne State)
Otto Szász	(Cincinnati)
John Todd	(London)
Olga Taussky-Todd	(London)

A complete roster of INA personnel for Summer 1948 and for other periods is given in Appendix F.

During this period D. R. Hartree gave a series of lectures pertinent to automatic computing machinery and to related problems in numerical analysis. By so doing he gave direction to the research to be carried out at INA. He envisioned a program centered around the problem of solving ordinary and partial differential equations by machine methods. Studies in numerical integration were carried out by D. R. Hartree, W. E. Milne, and R. E. Greenwood. A "Simpson's Rule" for the numerical integration of Wiener Integrals was devised by R. H. Cameron. E. F. Beckenbach was concerned with applications of conformal mapping and with the study of inequalities. H. A. Rademacher carried out numerical investigations of the zeros of the Zeta-function and studies in number theory. The Todds and O. Szász continued investigations begun earlier. This indicates the type of programs in which the senior staff were engaged during Summer 1948.

A problem completed by the Todds [106] while at INA was a problem arising in connection with copyright law. This is related to the theory of error-correcting codes, to the theory of factorial experiment designs, and to the practice of (British) football pools. Although the original problems were solved by hand, the next case was not solved until much later with the heavy use of a computer by H. J. L. Kamps and J. H. van Lint [41].

The Todds also introduced the study of the Hilbert matrices

$$\mathbf{H}_n = [1/(i+j-1)] \quad (i, j = 1, 2, \ldots, n)$$

for $n = 2, 3, \ldots$. These matrices are positive definite symmetric matrices whose inverses are given by explicit formulas. Their eigenvalues are clustered near $\lambda = 0$. These matrices are therefore highly ill-conditioned even for relatively small values of n. We found that Hilbert matrices were useful test matrices for testing numerical procedures involving matrix computations. A detailed account of Hilbert and related matrices is given in Chapter IX.

John Todd also analyzed the behavior of a 5-term difference approximation to the differential equation $y'' = -y$, which had a pathological behavior. This was an early example of the study of numerical stability in this area. He later analyzed the convergence and stability of various discretizations of the classical second order partial differential equations using Williamson's theorem on the characteristic values of certain partitioned matrices.

Otto Szász was one of the few visitors who spent several periods at INA. As remarked above, his specialty was the theory and applications of infinite series. Among his work at INA was the study of a generalization of Bernstein polynomials which give a construction of a uniform approximation to continuous functions by polynomials, of which an existence proof was given by Weierstrass. Szász also collaborated with John Todd in the study of the problem of numerical evaluation of improper integrals. Classical quadrature formulas, such as Simpson's Rule, apply to integrals over a finite interval $[a,b]$. The problem at issue is the approximate evaluation of the integral of $f(x)$ over $[0, \infty)$ as the limit of Riemann sums $\Sigma\, hf(nh)$. See his collected papers [100].

George E. Forsythe was the first regular member of research staff of INA. He came to INA with a strong background in the applications of mathematics to meteorology. He was a universalist in the sense that he collaborated with all members of the research staff and with other members of INA. He was very active in the computational aspects of the projects pursued at INA. He also became a leader in the educational program of the Institute. George Forsythe received his Ph. D. at Brown University in 1941. During 1941–42 he was an instructor of mathematics at Stanford University. During World War II he participated in the meteorology program at UCLA, at which time he collaborated with Jörgen Holmboe and William Gustin in writing a book entitled *Dynamic Meteorology*. During 1946–47 he was a Research Engineer at the Boeing Aircraft Company in Seattle. He returned to UCLA in 1947 as an Assistant Professor in Meteorology, before joining INA in 1948. His background in applied mathematics helped make him an outstanding member of the research group at INA.

As a part of its educational program INA employed junior members who were candidates for advanced degrees. They acted as assistants to the senior staff. The first appointees were Raymond P. Peterson, Jr., Robert H. Sehnert, and Marvin L. Stein.

One of the significant appointments made at this time was the addition of Everett C. Yowell to the computational unit to be in charge of Machine Computations. Being a mathematical astronomer he had considerable experience in this area. He quickly became the chief advisor to the research staff on the use of the CPC and other equipment. He also performed useful computational services for various governmental agencies.

Among the peripheral activities, which John Curtiss on occasion frowned upon, were studies in the theory of orthogonal polynomials. For instance, there were several proofs of Turán's inequality for Legendre polynomials

$$P_n^2 - P_{n-1}P_{n+1} \geqslant 0$$

and its generalizations by Szász, Beckenbach, Seidel, and Forsythe. A conjecture by John Todd on the extrema of Legendre Polynomials, established asymptotically by R. Cooper, was completely established by G. Szegö and then established in the Jacobi case by O. Szász and in the Laguerre case by J. Todd. Attempts to obtain a unified treatment have not succeeded so far. Detailed references to the papers mentioned in this paragraph are given in the second paper reproduced in Appendix B.

Such diversions from the main program are unavoidable. When a group of mathematicians learns of a problem within or near their field of competence, it is only natural for them to attack it by their own methods. These digressions improve communication between members of the organization who learn the expertise of their colleagues. A somewhat similar problem arises in Service Departments. Arrangements have to be made so that the senior people have some freedom for research in their field of interest and so that the junior people have educational opportunities to improve their skills, either at in-house courses and seminars or at local universities. These are mentioned in the course of this history. Generally arrangements were made to hire for Summer work winners of the Westinghouse Science Talent Search—a notable example is E. C. Dade at NBS-Washington. Later a program of post-doctoral research associates under the control of the National Research Council was instituted. Among those appointed at NBS-Washington, who made particularly relevant contributions to numerical mathematics, were Marvin Marcus and John R. Rice.

CHAPTER III

THE PERIOD FALL 1948 THROUGH SPRING 1949

At the end of the Summer Session, several of the investigators returned to their respective universities. The Todds returned to the University of London. A year later they joined the staff of the National Applied Mathematics Laboratories at NBS in Washington, DC. John Todd became Chief of the Computation Laboratory and Olga Taussky-Todd became a Mathematics Consultant in the Division. They had an active interest in the development of INA and made frequent visits to INA.

Hartree returned to Cambridge, England, at the end of October. As a result John Curtiss became Acting Director of INA until a qualified director could be found. The Senior Researchers during the period October 1948–June 1949 were:

Edwin F. Beckenbach	(UCLA Representative)
Gertrude Blanch	(Head of Computational Unit)
George E. Forsythe	(INA)
Samuel Herrick	(UCLA)
Harry D. Huskey	(Head of Machine Development)
Cornelius Lanczos	(Joined INA in January 1949)
William E. Milne	(Oregon)
Alexander M. Ostrowski	(University of Basel, Switzerland)
Wladimir Seidel	(Wayne State)

The Junior Members of the Research Staff for this period were:

Howard W. Luchsinger	Raymond P. Peterson, Jr.
Robert H. Sehnert	Marvin L. Stein

As the Institute developed, Wilbur W. Bolton, Jr., joined the administrative unit to help with budget problems. He was the NBS budget chief, 1951–56, and later became the Budget Officer of the National Science Foundation.

In December 1948 Harry D. Huskey arrived at INA for the purpose of designing and building an Electronic Digital Computer. He received his Ph. D. in mathematics in 1943 under the sponsorship of Tibor Rado at Ohio State University. He was an instructor in mathematics at Ohio State, 1942–43, and at the University of Pennsylvania, 1943–46. He first became interested in computers in 1944 working on the ENIAC project at the University of Pennsylvania. The ENIAC was the first large scale electronic computer to be constructed. The year 1947 was spent in England at the National Physical Laboratory where he worked with Turing on the ACE computer project, supervised the design, and began construction of a pilot model of the ACE. In 1948 Huskey joined the staff of the National Bureau of Standards in Washington, DC, as Chief of the Machine Development Laboratory of the Applied Mathematics Division. In December 1948 he was transferred to INA as an Associate Director in charge of the design and construction of the SWAC—the National Bureau of Standards Western Automatic Computer. Huskey and his associates were pioneers in this field. The story of the SWAC is given in two articles by Harry Huskey [37,38]. We have reproduced the second of these articles in Appendix C to give the reader an authoritative description of the SWAC Project.

In January 1949 the UCLA physicists, Alfredo Baños, Jr. and David S. Saxon became consultants to INA. They made significant contributions to the program at INA.

Cornelius Lanczos became the second regular member of the research staff at INA. He was born in Hungary in 1893. In 1921 he obtained his Ph. D. in Physics for a thesis entitled "Function

Theoretical Relations of Maxwell's Vacuum Equations" at the University of Szeged under Professor Rudolph Ortvay. Shortly thereafter he went to Germany, first to the University of Freiburg in Breisgau and then to Frankfurt am Main. In 1926 he published a paper on an interpretation of quantum mechanics on a continuum basis in terms of integrals. This was the earliest continuum theoretic formalism of quantum mechanics, preceding Schrödinger's first communication by about 4 weeks. Lanczos was also interested in the Einsteinian equation of gravity, and, at the invitation of Einstein, worked with Einstein on the theory of relativity at the University of Berlin during the Academic year 1928–29. In 1931 he published his first paper on the quadratic action principle in relativity. Lanczos came to the United States in 1931. He joined the staff at Purdue University where he remained for 15 years. He became interested in numerical analysis and, in 1938, developed his theory of economization of polynomials, now known as the Lanczos-Tau method. In 1942 two papers were published in conjunction with G. C. Danielson on practical techniques of Fourier Analysis. In 1943–44 Lanczos was on the staff of NBS working on the Mathematical Tables Project. During the period 1946–48 he was employed by Boeing Aircraft Company in Seattle, where he obtained industrial experience in numerical methods. Although initially his numerical methods were designed for use on standard desk computers, they turned out to be very suitable for "automatic" machine computation. His wide experience enabled Lanczos to make outstanding contributions at INA.

During this period the research program was channeled into the following three lines of work: development of numerical methods in conformal mapping, the exposition and further development of the numerical solution of ordinary differential equations, and studies in the use of probabilistic methods in numerical analysis. In addition, certain studies, such as the programs begun earlier by Herrick and by Beckenbach, were continued. Wladimir Seidel of Wayne State University initiated a study on numerical methods in conformal mapping by compiling a bibliography of pertinent references. He also translated into English a collection of Russian papers on conformal mapping entitled, "Conformal representation of simply and multiply connected regions," by L. Kantorovitch, V. Kryloff, G. Golouzin, P. Melentief, and others. William E. Milne, a well-known numerical analyst from the University of Oregon, undertook the preparation of a monograph [63] on numerical solutions of differential equations. Lanczos sought to develop a practical and economical method of evaluating the eigenvalues of complex matrices and to develop an economical method for solving simultaneous linear equations. Raymond P. Peterson, Jr. and John Curtiss began the study of Monte Carlo methods for solving partial differential equations and integro-differential equations.

Alexander Ostrowski of the University of Basel, Switzerland, became a frequent visitor at INA. We always looked forward to his visits. During his first visit he developed various formulas for evaluating definite integrals and sought to find new and better ways of dealing with algebraic equations. He also participated in the research program on numerical methods in conformal mapping. He was a catalyst in the group.

Ostrowski was one of the most versatile mathematicians of our time. This is evident from his Collected Papers being published in six volumes, beginning in 1983, by Birkhäuser. He is perhaps best known for his determination of the valuations of the field of rationals. His appointment to NBS-INA was suggested by the Todds, who were aware of his earlier work on the stability of the Newton Process and what is now called complexity. It was he who introduced the "Horner" as a unit of work: the evaluation of a polynomial of degree n by the Horner process. He later showed that the Horner method (synthetic division) was (in cases $n = 3,4$) the most efficient way of evaluating a polynomial. He also began the study of symbolic integration. Ostrowski published in 1937 two classical papers concerning bounds for determinants, connected with the results of Hadamard, Minkowski, and others. Following a suggestion made to him at INA by Olga Taussky, he made many contributions to Gerschgorin Theory and to obtaining bounds of eigenvalues of matrices. He was also concerned with iterative methods for the solution of linear systems. His attention was invited by John Todd to a popular method for the solution of the algebraic eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$. This method consists of guessing an estimate of $\mathbf{v}$ of an eigenvector of $\mathbf{A}$, obtaining an estimate of the eigenvalue λ by the Rayleigh quotient of $\mathbf{v}$, and obtaining a new estimate by solving the nearly singular system $\mathbf{Ax} = \lambda\mathbf{x}$. Ostrowski analyzed this process in a series of six papers and said that only his training in analytic number theory in the Landau tradition enabled him to complete the study. For a survey of his work in Linear Algebra, see the Preface by his student, Walter Gautschi, to *Linear Algebra and Applications,* Vols. 52–53, dedicated to him on his 90th birthday in 1983. The first edition of his book [68] *Solutions of Equations and Systems of Equations* was based on his 1952 lectures at NBS.

As a public service John Curtiss initiated a Colloquium Series open to the public. A speaker was free to choose his own topic. Many of the speakers were from INA. Mainly they described the activities in which they were engaged. Other speakers were visitors of the Institute who spoke on topics of their own interests. During this period the following visitors gave colloquium lectures:

A. S. Besicovitch	G. Brown
C. F. Davis	H. H. Germond
J. J. Gilvary	C. Hastings
Casimir Kuratowski	Sir Harold Spencer Jones

A "Symposium on the Construction and Applications of Conformal Maps" and a "Symposium on the Monte Carlo Method" were held at INA at the end of June 1949. Sandwiched between these two symposia was a 2-day condensed course in automatic computation given under the direction of Harry Huskey. The combined registration for all events totaled about 350 persons.

The purposes of the "Symposium on the Construction and Applications of Conformal Maps" (June 22–25) were to consider physical applications of conformal maps and their generalization, and to study the construction of conformal maps with a view to determining the possible applicability of high-speed electronic computing machines in this direction. This symposium was arranged by a committee consisting of E. F. Beckenbach (chairman), C. Lanczos, A. Ostrowski, and W. Seidel. The Proceedings of this Symposium was edited by E. F. Beckenbach and was published as NBS Applied Mathematics Series 18. The title page and table of contents are listed in Appendix D. The program of the Symposium can be found by looking at this table of contents.

The "Symposium on the Monte Carlo Method" was jointly sponsored by the Rand Corporation and the Institute for Numerical Analysis with the cooperation of the Oak Ridge National Laboratory. The committee on arrangements consisted of J. H. Curtiss, H. H. Germond, A. S. Householder, C. C. Hurd, and R. P. Peterson. The Symposium was held on June 29, 30, and July 1, 1949.

The Monte Carlo method can be described quite generally as the representation of a physical or mathematical system by a sampling operation whose expectation or variance gives the behavior of the system under scrutiny. Thus, for example, the numerical integration of partial differential equations of a certain type can be accomplished by building up a large sample of trials of certain stochastic processes whose probability functions asymptotically satisfy the partial equations. In certain physical situations formally represented by such equations, the physicist may prefer to place primary emphasis on the stochastic processes and the associated sampling operations, which he will regard as new mathematical models to be used in place of the continuous models of classical applied mathematics. J. von Neumann and S. M. Ulam are credited as being the originators of Monte Carlo methods. John Curtiss was a strong proponent of these methods and initiated a program of studies of Monte Carlo methods at INA. In his IBM paper (see Appendix B) he discussed the distinguished pedigree of Monte Carlo methods. Under his leadership, INA pioneered in certain applications of Monte Carlo methods, especially in finding eigenvalues of Schrödinger's equation. INA also was concerned with developing methods for generating random numbers. The generation of pseudo-random numbers with uniform distribution by number-theoretical methods was studied by D. H. Lehmer and Olga Todd. Forsythe and others also considered the case of other distributions. Testing of the quality of these pseudo-random numbers was done in collaboration with the Statistical Engineering Laboratory of the AMD. John Todd made various controlled experiments on the Monte Carlo solution of the Dirichlet problem in various dimensions.

Unfortunately, today the expression "Monte Carlo Method" is loosely applied to any empirical sampling solution of a probability or statistical problem, a much older and less sophisticated technique (see D. Teichroew [112]).

The purposes of this Symposium were to interchange information concerning the usefulness of the method, to stimulate discussions relative to its limitations, and to indicate directions in which further theoretical research is needed. The program of the Symposium can be found in the Proceedings of this Symposium, published as NBS Applied Mathematics Series 12 in 1951, edited by A. S. Householder, G. E. Forsythe, and H. H. Germond. The title page and table of contents are reproduced in Appendix D.

We mention here the later (1956) Gainesville Symposium on Monte Carlo Methods to which INA contributions were made by J. H. Curtiss, T. S. Motzkin, O. Taussky-Todd and J. Todd. It was supported by the Air Force and the Proceedings [130] were published by Wiley.

Cooperation with other scientific organizations was a central part of administrative policy of INA. In the Spring of 1949 a Seminar on Stochastic Processes was established jointly with the Rand Corporation as part of the Monte Carlo project of INA. The seminar met regularly every Thursday, alternating between the Institute building and the Rand building in Santa Monica. It was continued during the summer under the direction of W. Feller.

A second cooperative venture was a conference on establishing a west coast mathematical journal. This conference was held on June 27, 1949. Some 50 or 60 west coast mathematicians were invited to the conference by John Curtiss, the Acting Chief of the Institute. They represented all the west coast universities, the Rand Corporation, and the National Bureau of Standards. A. W. Tucker, of Princeton University, was chairman of the conference, and, in addition, served as an official representative of the American Mathematical Society. A proposal of the National Bureau of Standards to establish a new journal under cooperative editorship, with publication by the Government Printing Office, was placed before the group. After a lengthy discussion it was decided that a journal should be established under the sponsorship of the universities on the west coast with the cooperation of the Institute for Numerical Analysis. An exploratory committee was formed under the chairmanship of E. F. Beckenbach. This led to the establishment of the *Pacific Journal of Mathematics* under the sponsorship of the west coast universities and INA. The headquarters of the *Pacific Journal* was located at INA-UCLA. Beckenbach became its first Editor-in-Chief. He employed Elaine Barth to be his Administrative Assistant. Initially, she was in charge of preparing the manuscripts for photo offset. At present she is still responsible for the preparation of all manuscripts to be published in the *Pacific Journal of Mathematics.*

Another journal in which NBS took an important part was *Mathematical Tables and other Aids to Computation,* originally established in 1943 by the National Research Council. In 1960 its name was changed to *Mathematics of Computation.* Publication was taken over by the American Mathematical Society in part in 1962 and wholly in 1966. Among those connected with NBS-INA, who served as editors, were E. W. Cannon, D. H. Lehmer, J. Todd, and C. B. Tompkins.

The NBS has its own *Journal of Research* and many important papers written by INA members are published there. Curtiss insisted that this should not be a vehicle for publishing papers not acceptable in outside journals. The only deviation from this policy was to cover cases of papers involving tables and/or experimental computing.

The early tables of the Mathematical Tables Project were published by the Government Printing Office, beginning in 1939. A later series was published commercially by Columbia University Press. In 1948 John Curtiss organized the Applied Mathematics Series. The series now includes 63 volumes, including tables, handbooks, symposium proceedings, and lecture notes. The volumes, particularly relevant to NBS-INA, were 12, 15, 18, 29, 39, 42, 49, and 55. They are referred to in the text and in the appendices. AMS 55 was particularly successful. It is a *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun and published in 1964 by the Government Printing Office (later also by Dover and the 1972 edition was reprinted in 1984 by Wiley). G. Blanch, U. Hochstrasser, and T. H. Southard of INA contributed chapters in this book. C. B. Tompkins and J. Todd were on the National Research Council Committee, chaired by P. M. Morse, which provided technical guidance on this volume.

It is appropriate to mention here the NBS-INA contributions to the *Handbook of Physics* which was conceived by E. U. Condon when he was Director of NBS. Originally planned as an NBS publication, it was actually published commercially by McGraw-Hill in 1964 (with a second edition in 1967) with Hugh Odishaw as co-editor. The writing of Part I: Mathematics, was assigned to Olga Taussky-Todd. She wrote chapters on Algebra, Ordinary Differential Equations, and Operators. M. Abramowitz wrote on Integral Equations, F. L. Alt on Fundamentals, J. Todd on Analysis, F. John on Partial Differential Equations, A. J. Hoffman on Geometry, E. U. Condon on Vector Analysis, C. Lanczos on Tensor Calculus, C. B. Tompkins on Calculus of Variations, C. Eisenhart and M. Zelen on Elements of Probability, and W. J. Youden on Statistical Design of Experiments. In the second edition, the chapters on Operators and on Integral Equations were absorbed in a new chapter by J. L. B. Cooper. A new chapter on Numerical Analysis by J. Todd was added.

As part of its educational program, the Institute undertook an experimental program for graduate students during Summer 1949 at the suggestion of H. F. Bohnenblust and Morgan Ward of Caltech. The participants were:

Harold Gruen	(UCLA)
Robert J. Diamond	(Caltech)
Robert C. Douthitt	(UCB)
Ernest S. Elyash	(Cornell)
Hans F. Weinberger	(Carnegie Tech)
James G. C. Templeton	(Princeton)
Lloyd K. Jackson	(UCLA)

Special courses were arranged for these students.

CHAPTER IV

THE PERIOD SUMMER 1949 THROUGH SUMMER 1950

John Curtiss continued as Acting Director of INA during Summer 1949. Early in September, J. Barkley Rosser became Director of the Institute. The members of the Senior Research Staff for Summer 1949–Spring 1950 were:

J. Barkley Rosser	(Director, Cornell)
Forman S. Acton	(INA)
Lars V. Ahlfors	(Harvard, Summer 1949)
Edwin F. Beckenbach	(UCLA, Summer 1949)
Gertrude Blanch	(INA)
Monroe D. Donsker	(Cornell, Summer 1949)
Aryeh Dvoretzky	(Hebrew University, Spring 1950)
William Feller	(Cornell, Summer 1949)
George E. Forsythe	(INA)
Samuel Herrick	(UCLA, Summer 1949)
Magnus R. Hestenes	(UCLA Representative)
Harry D. Huskey	(INA)
Mark Kac	(Cornell)
William Karush	(Chicago)
Cornelius Lanczos	(INA)
Alexander M. Ostrowski	(University of Basel, Switzerland)
Otto Szász	(Cincinnati)
Stephen E. Warschawski	(Minnesota, Summer 1949)
Wolfgang R. Wasow	(INA)

Alfredo Baños and David S. Saxon of the UCLA Physics Department continued to serve as consultants at INA.

The members of the Junior Research Staff were:

George Gourrich	Harold Gruen
Lloyd K. Jackson	Harold Luxenberg
Raymond P. Peterson, Jr.	Robert Sehnert
Marvin L. Stein	

In Spring 1950 they were joined by Richard E. Cutkosky and Stuart L. Fletcher. Upon receiving his Ph. D., Raymond P. Peterson became a member of the Senior Staff.

The following visitors at INA gave Colloquium Lectures at INA during this period:

E. T. Benedict	M. Fekete
R. P. Feynman	R. A. Fisher
O. Helmer	J. O. Hirschfelder
R. Isaacs	I. Kaplansky
S. Lefschetz	E. Penney
W. Prager	G. Szegö
A. W. Tucker	

Wolfgang Wasow became the third regular member of the research staff concerned primarily with research. He was a specialist in ordinary and partial differential equations. He approached these equations not only from the classical point of view but also through the use of Monte Carlo methods. He became interested in Monte Carlo methods due to a suggestion made by John Curtiss. (See [124–126].) Although he had no previous experience in numerical methods for solving differential equations, he learned quickly and soon became an expert in this field. Wasow received his Ph. D. from New York University in 1942.

During Summer 1949 considerable progress was made on the problem of numerically constructing conformal maps and determining eigenvalues of linear operators. Ahlfors proposed two numerical methods of constructing the mapping function which maps a polygon on a circle or half-plane. Ostrowski weakened the restrictions on the applicability of the Theodorsen-Garrick method of constructing conformal maps. Warschawski made a comprehensive review of results dealing with the variation of the mapping function corresponding to deformation in the mapped region. Beckenbach worked full time on the assembling and editing of the "Proceedings of the Symposium on Conformal Mapping" [5].

The Monte Carlo method was demonstrated for the first time to be a useful tool in solving eigenvalue problems. Mark Kac and M. D. Donsker developed a method for finding the lowest eigenvalue and the corresponding eigenfunction of the Schrödinger equation by the Monte Carlo technique. Feller and Forsythe also participated in this program. Lanczos continued his studies on his special methods for dealing with eigenvalue problems. Hestenes, Karush, and Stein pursued the problem of obtaining numerical solutions of optimization problems. Herrick and Szász continued the programs that they introduced earlier.

Considerable numerical experiments were made to test theoretical results even though the high-speed computer was not yet available. P. L. Morton and D. H. Lehmer of Berkeley visited INA and gave a favorable report of the progress made in the construction of the SWAC.

As remarked earlier, J. Barkley Rosser became the Director of INA in September 1949. Although his early training at the University of Florida was in physics, he received his Ph. D. at Princeton in 1934 in mathematical logic. He was a Proctor Fellow at Princeton, 1933–35, and a National Research Council Fellow at Harvard, 1935–36. In 1936 he joined the Department of Mathematics at Cornell University. During 1944–46 he was chief of the Theoretical Ballistics Section at Allegany Ballistics Laboratory in West Virginia. He served as a consultant for the Applied Physics Laboratory, Johns Hopkins University, 1945–1963. Rosser was well versed in both pure and applied mathematics. He was an expert in logic, analytic theory of numbers, infinite series, and classical analysis. He had done a lot of computation (hand) in connection with Fermat's Last Theorem and in connection with the tabulation of functions related to the error-function that turned up in ballistics. Rosser was exceptionally well qualified to be the Director of INA. Although Rosser served as Director of INA for only one year, his impact on INA was great. The programs he initiated were basic and continued to be an important part of the research program at INA. As noted below, he returned to INA for Summer 1951 at which time he participated in a symposium on numerical algebra. In 1953 Rosser served on an evaluation committee of the National Bureau of Standards (see Chapter VII). In addition he served on an advisory panel for NBS, 1962–1968. During the period 1959–61, Rosser was the Director of the Communications Research Division of the Institute for Defense Analyses. He served as Chairman of the Mathematics Division of the National Research Council, 1960–62, and was a member of the Conference Board of the Mathematical Sciences, 1962–64. In 1964–66, Rosser was a member of the Space Technology Panel of the President's Science Advisory Committee. In 1963 Rosser left Cornell to accept the directorship of the Mathematics Research Center at the University of Wisconsin, where he served with distinction. He was always active in various professional organizations and served as president of the Society for Industrial and Applied Mathematics (SIAM) and of the Association for Symbolic Logic.

During this period special emphasis was placed on iterative methods for solving systems of linear equations and for finding eigenvalues of matrices. To this end, Rosser instituted a weekly seminar concerned primarily with these topics. This seminar was attended principally by Rosser, Forsythe, Hestenes, Karush, Lanczos, and Stein. Rosser and Forsythe were chiefly responsible for the study of systems of linear equations. Forsythe, in particular, undertook the task of classifying the various known methods for solving systems of linear equations. Hestenes, Karush, and Stein were responsible for the study of methods for finding eigenvalues of matrices with particular emphasis on gradient methods, power methods, and inverse power methods. Lanczos continued the study of his orthogonalization techniques for finding eigenvalues of matrices and for solving linear systems. He also devised techniques for using Chebyshev and other orthogonal polynomials to obtain good estimates of their solutions.

Of interest was a numerical experiment on finding the eigenvalues and eigenvectors of a symmetric matrix with close eigenvalues. The purpose was to test the methods devised by Lanczos and by Hestenes and Karush. Rosser constructed an 8×8 symmetric matrix with close eigenvalues. Five were bunched near 1000. Three of these were very close. The other two were equal. The negative of one of these was also an eigenvalue. One of the two remaining eigenvalues was zero and the other was very close to zero. Each eigenvector was computed to a 10-digit accuracy. The two methods were equally successful. Time comparisons for the two methods could not be made because the Lanczos method was carried out on a desk computer while the method of Hestenes and Karush was carried out on an IBM CPC. Considerable time was spent in developing techniques for separating close eigenvalues. After developing a suitable technique, it took about 11 hours to solve the problem on the CPC by the gradient method developed by Hestenes and Karush. At the present time these computations could be carried out in a few minutes.

Another interesting computational episode occurred during this period. A group in industry was having difficulties in inverting a certain 10×10 matrix. They asked INA for help. We too had difficulties although our computations were carried out under the supervision of G. Blanch, a nationally recognized expert in Numerical Analysis. The computations were made on a desk calculator using a standard elimination method with pivoting. After spending too much time and money on the project, Rosser inquired about the origin of this matrix. He found that it was a scaled orthogonal matrix. Accordingly, its inverse could be obtained by a suitable rescaling of the transpose of the matrix, an operation that required very little computation. It should be pointed out that even finding the inverse of an orthogonal matrix by an elimination procedure can require lots of pivoting and that, on a desk calculator, pivoting can be a very time consuming operation. This episode shows that matrix inversion is not a trivial matter even with a "good" matrix.

Research programs started previously were continued. Studies on the Monte Carlo method were carried out by Forsythe, Dvoretzky, Kac, and Wasow. Improved methods for generating random numbers were devised by Forsythe and Rosser. Lanczos devised a new method for inverting Laplace Transforms with applications to network analysis. Methods for improving rates of convergence of series were considered by Rosser and Szász.

SUMMER 1950

As before, the research staff was enlarged during the Summer Session. The following researchers joined the Senior Staff for Summer 1950:

Paul Erdös	(Hungary)
Richard P. Feynman	(Cornell, IAS)
Fritz John	(New York)
Edward J. McShane	(Virginia, IAS)
William E. Milne	(Oregon)
Theodore S. Motzkin	(INA)
Raymond P. Peterson, Jr.	(INA)
David S. Saxon	(UCLA)

The new Graduate Fellows for this period were:

Harold P. Edmundson	Robert K. Golden
William G. Hoffman	

In addition Lloyd Jackson participated in the program of Conformal Mapping. Upon obtaining the Ph. D. he accepted a position at the University of Nebraska.

The following visitors of INA gave colloquium lectures during Summer 1950:

A. A. Albert	G. R. Boulanger
J. Charney	L. Fox
J. C. P. Miller	A. Weinstein

Feynman and McShane were supported by the Institute for Advanced Study. They were a stimulus to the group.

David Saxon, a member of the UCLA Physics Department, was an important addition to the staff. He strengthened our program on applications to physics. He also served as a consultant to the mathematical research staff, thereby giving guidance in the choice of problems of interest to physicists. He later became President of the University of California.

The addition of Fritz John strengthened our program on numerical solutions of partial differential equations. When Rosser returned to Cornell in September, John became the Director of INA.

Theodore S. Motzkin was a new addition to the regular research staff at INA. He was a very versatile and knowledgeable mathematician, who specialized in combinatorics, linear inequalities, linear programming, and approximation theory. He started university study when not yet 16, and studied at Göttingen and Berlin, and finally at Basel where he studied and collaborated with A. Ostrowski. His thesis, under Ostrowski, finished in 1934, developed a theory of linear inequalities. He is credited with being one of the originators of the theory of linear inequalities and linear programming. His first academic position was at Hebrew University in Jerusalem from 1935 to 1948. During World War II he was a cryptographer for the British Government in Palestine. Motzkin came to the United States in 1948. After 2 years at Harvard and Boston College, he came to NBS and UCLA. He collaborated with visitors and with other members of the research staff and, in particular, with Forsythe and Wasow. He was a valuable addition to our research staff. An early contribution of Motzkin deserves mention. We have already described Ostrowski's work on optimal character of the Horner process for the evaluation of a polynomial $p(x)$ for a single value of x. Suppose instead we want to evaluate $p(x)$ for many values of x. Is it possible to improve on the n multiplications by a preliminary investment? Motzkin showed that, surprisingly, about $n/2$ multiplications would suffice. Essentially this depends on an algebraic identity which represents a polynomial of degree n as a polynomial of degree of about $n/2$ in another variable. Motzkin also studied, often in collaboration with J. L. Walsh of Harvard, a variety of problems in polynomial approximation. His *Selected Papers* have been published [67].

The U.S. Air Force, towards the end of World War II, set up a project "Scientific Computation of Optimal Programs" to reduce the time taken to "plan programs." Experimental calculations, theoretic investigation and the specification and procurement of appropriate computers were in progress at NBS Washington. Program Planning *was* the title of the INA Project (P&P April/June 1950) organized by Motzkin. The main activity at INA was concerned with Linear Programming and methods to solve linear programs, including the Simplex Method. For historical accounts of this see the publications of George B. Dantzig, in particular his articles in AMS 15 and in [131].

M. Kac suggested to Olga Todd the study of pairs of matrices A, B with the L-property, that is, pairs A, B such that the characteristic roots of $a\mathrm{A}+b\mathrm{B}$ are $a\lambda_i+b\mu_i$, where $\{\lambda_i\}$ and $\{\mu_i\}$ are the characteristic roots of A and B in some order, independent of a and b. Pairs of matrices of this type had arisen in a statistical context connected with a theorem discovered by C. C. Craig and studied by H. Hotelling of which Olga Todd [107] gave a new proof, using the L-property. A fruitful collaboration between Olga Todd and Motzkin developed [66]. One result was a new basic theorem in perturbation theory: "If every $aA+bB$ is diagonalizable for all ratios a/b (including ∞) except perhaps for one value, then the pair A, B has the L-property." This result is the best possible. Further if there is no exceptional ratio, then A and B commute. Another proof of this theorem was given by Tosio Kato in his books [44,45] on Perturbation Theory for Linear Operators. H. Wielandt also investigated the L-property.

Related to this work is the characterization of normal matrices by A. J. Hoffman and Olga Todd [34]. Various problems connected with commutativity and its generalizations were studied by Olga Todd and Kato [48].

Perhaps the most important event that occurred during this period, was the dedication, on August 17, 1950, of the SWAC, the National Bureau of Standards Western Automatic Computer. This machine was sponsored by the Office of Air Research of the Air Material Command, USAF. It was designed and built by Harry Huskey and his staff at INA. The speakers at the dedication ceremony were E. U. Condon, Director of NBS; Colonel F. S. Seiler, Chief of the Office of Air Research, Department of the Air Force; L. N. Ridenour, Dean of the Graduate School of the University of Illinois; J. H. Curtiss, Chief of the National Applied Mathematics Laboratories; and H. D. Huskey, Chief of the INA Machine Development Unit. The dedication was followed, on August 18, by a symposium on applications of digital computing machines to scientific problems. A description of this symposium is given in Appendix D.

At the time of the dedication of the SWAC the members of the Machine Development Unit were:

Harry D. Huskey	Biagio F. Ambrosio
Edward L. Lacey	David F. Rutland
Roselyn S. Lipkis	Harold Luxenberg
Brent H. Alford	Harry T. Larson
Arnold Dolmatz	Michael J. Markakis
Blanche C. Eidem	John L. Newberger
Sidney S. Green	James W. Walsh

These and other temporary personnel were responsible for the building of the SWAC. Among those who made important contributions later were Ragnar Thorensen and E. W. Cannon, who was Assistant Chief of AMD for computer development and was in residence at INA during a critical period in the development of SWAC.

The research program for Summer 1950 was, in the main, a continuation of the programs started earlier. R. Fortet of the University of Paris joined Dvoretzky, Kac, and Wasow in the study of Monte Carlo methods for solving certain partial differential equations. Milne resumed work on his monograph on numerical solutions of differential equations. About one-third of the book was devoted to the numerical solution of partial differential equations. Forsythe, Hestenes, Karush, Lanczos, Rosser, and Stein continued their studies on finding eigenvalues and solving linear systems. In particular, Rosser developed a method for the exact solution of linear systems with integral coefficients. Rosser also studied the acceleration of the convergence of slowly converging series.

CHAPTER V

THE PERIOD FALL 1950 THROUGH SPRING 1951

Fritz John became the Director of INA in September 1950. Fritz John received his Ph. D. at the University of Göttingen in 1933, specializing in the Calculus of Variations. He was a Research Scholar at Cambridge University, 1934–35. John was a member of the Department of Mathematics at the University of Kentucky, 1935–42. During World War II, he served as a Mathematician at the U.S. War Department. In 1946 he joined the staff at the Courant Institute, New York University. Fritz John is an expert on Partial Differential Equations, Nonlinear Elasticity, Analysis, and Geometry. His wide background enabled him to make significant contributions to our program on Differential Equations at INA. John served as Director of INA for 1 year. Upon returning to the Courant Institute, he continued to make significant contributions and received many honors, including the George David Birkhoff Prize in Applied Mathematics and membership in the National Academy of Science. He gave the Gibbs Lecture of the American Mathematical Society in 1975. See John [40]. His collected papers have been published by Birkhäuser, Boston in 1985.

Senior Research Personnel for this period were:

Fritz John	(Director, NYU)
Forman S. Acton	(INA)
Gertrude Blanch	(INA)
Milton Dandrell	(UCLA)
Aryeh Dvoretzky	(Hebrew University)
George E. Forsythe	(INA)
Robert Fortet	(Paris, France)
John W. Green	(UCLA)
Magnus R. Hestenes	(UCLA Liaison)
Harry D. Huskey	(INA)
William Karush	(Chicago)
Cornelius Lanczos	(INA)
Hans Lewy	(UC, Berkeley)
William E. Milne	(Oregon)
Theodore S. Motzkin	(INA)
Lowell J. Paige	(UCLA)
David S. Saxon	(UCLA)
Wolfgang R. Wasow	(INA)

Members of the Junior Research Staff were:

Harold P. Edmundson	Robert K. Golden
Harold Gruen	Robert M. Hayes
William C. Hoffman	Marvin L. Stein
James P. Wesley	

When Stein received his Ph. D. in January he became a member of the senior research staff. In his Ph. D. thesis, Stein justified the use of gradient methods for variational problems and for self adjoint boundary value problems.

Colloquium lectures were given by the following visitors of INA:

N. Aronszajn	A. Erdélyi
E. Gerjuoy	K. Knopp
G. Pólya	S. Sherman
J. J. Stoker	D. van Dantzig

Dvoretzky, Fortet, Karush, and Lewy returned to their universities at the end of Fall 1950. Hestenes served only in a limited capacity. He was in charge of the contract that NBS had with UCLA. Paige, an algebraist from UCLA, joined the group in January.

The weekly seminar initiated by Rosser and now led by John turned its attention to the study of numerical methods for solving partial differential equations. Fritz John discussed the convergence of solutions of a finite-difference equation which approximated a parabolic partial differential equation with some non-linear terms, as the mesh size tended to zero while keeping a fixed "shape." As was often the case at INA, numerical studies were made in parallel to the theoretical ones; in particular in this connection G. Blanch completed a substantial contribution. Monte Carlo methods continued to be pursued by Dvoretzky, Fortet, and Wasow. More classical approaches were emphasized by John, Green, Lewy, and Milne. Forsythe participated in both programs. He also made numerical experiments on various schemes for solving systems of linear equations. He found that, in ill-conditioned cases, the optimal gradient method tended to "bog down." This suggested that an acceleration method was needed. Forsythe and Motzkin suggested one acceleration scheme and Hestenes suggested a simpler one. Considerable improvements were gained in the gradient method and in other iterative methods by the use of these acceleration schemes. These experiments eventually led to the formulation, by Hestenes, of the conjugate gradient method, a method which will be discussed later. M. L. Stein found that acceleration schemes were useful also in calculating eigenvalues of matrices. Experiments also showed that high precision arithmetic was needed for the preservation of significant figures in the matrix computations.

Research on the theory of Program Planning expanded considerably due to the activities of Motzkin, who invented some promising new methods for solving systems of inequalities, and through a special seminar sponsored jointly by INA and Rand.

So far we have been concerned mainly with the activities of the research staff. From the beginning, the computational unit under Blanch and Yowell was actively engaged in solving computational problems for the research staff, for various governmental agencies, and for contractors of the Federal Government. In addition the computational facilities of INA were made available to departments of west coast universities. In particular, the Departments of Astronomy, Chemistry, Geophysics, Meteorology, Physics, and Psychology at UCLA made full use of these facilities. They were among the first to carry out research in which machine computations played a significant role. By 1951 the demands for computational assistance were so great that it was difficult for the Computational Unit to fulfill its obligations. Accordingly, a new unit of INA, called the Mathematical Services Unit, was formed under the supervision of Harry Huskey. It was funded by the United States Air Force. One of the purposes of this unit was to encourage Federal Government contractors to learn how to use electronic computers. Accordingly, computational services using SWAC were made available to them. Many of these contractors made use of this service. Effectively, the NBS offer to these contractors was to augment their contracts by providing free computational services of a type which was not as yet available elsewhere. It is interesting to note that when IBM announced that they would build a "Defense Calculator" if they could get 12 orders, 6 of the 12 orders came from INA's computer customers. The "Defense Calculator" became the IBM 701—their entry into the "Electronic Computer Age."

At the end of June 1951 the members of the Mathematical Services Unit were:

Arnold D. Hestenes (Head)	Frederick H. Hollander
Marvin Howard	Roselyn S. Lipkis
Harold Luxenberg	Robert R. Reynolds
Thomas H. Southard	Everett C. Yowell

As noted earlier, this group formed an effective liaison between INA and various government agencies and government contractors.

A. D. Hestenes was the chief liaison officer. He was charged with the administration of the funds assigned to the Services Unit. He was responsible for "educating" various government agencies and contractors of these agencies on the facilities available to them at INA and on how electronic computing machines could be used to solve some of their problems.

Initially, Southard was on a sabbatical leave from Wayne State University. He found the activities at INA so challenging that he decided to accept a regular position at INA. Primarily, his duties were to do the "numerical" analysis part of the large variety of problems submitted to INA for solution. He was actively engaged in the educational functions of INA and, from time to time, taught numerical analysis at UCLA and at UCLA-Extension. Southard was interested in getting people together and organized social activities for members of INA and friends. Later he helped organize a National Meeting of the Association for Computing Machinery. He became involved in organizing the first chapter of SIAM (the Society for Industrial and Applied Mathematics) west of the Mississippi, and subsequently served as National Vice President (1954–6) and National President (1956–8).

Reynolds also did the "numerical analysis" part of problems to be solved at INA. In addition he wrote an extensive research paper on the Numerical Integration of the Rolling Pullout Equations for an Airplane.

Throughout the existence of INA, Yowell was in charge of machine computations and was a principal advisor to users of machines. He also wrote several research papers jointly with members of the research group.

At the request of the Office of Air Research, a 3-month training program for 18 Air Force Cadets was initiated. The purpose was to train USAF airmen in computing methods, the logical design of high-speed digital computers, and in coding and programming techniques for the SWAC, IBM, and hand machines. This course was taught by Acton, Reynolds, Milne, Luxenberg, Lipkis, and Yowell. This was part of the educational program sponsored by INA. From time to time members of the staff at INA gave courses in numerical analysis and in computational techniques at UCLA and UCLA-Extension. Graduate courses and seminars in fields of their own interests were given by:

J. H. Curtiss	P. Erdös
W. Feller	R. P. Feynman
G. E. Forsythe	H. D. Huskey
F. John	M. Kac
C. Lanczos	T. S. Motzkin
A. Ostrowski	H. A. Rademacher
J. B. Rosser	D. S. Saxon
I. J. Schoenberg	C. B. Tompkins
J. van der Corput	W. R. Wasow

The Mathematical Services section was responsible for a very large variety of projects. This is evidenced by the following sample of the projects which were on their "books" in the Quarterly Report, April–June 1951. Not all of these projects were active at the same time. Many other projects were attacked later.

Computing services for the research staff
The determination of the periods and amplitudes of the light variations of certain stars
Numerical studies of a non-linear parabolic differential equation
Mathieu Functions II
Special tables of Bessel functions
Tables for rocket and comet orbits
Punched card library
Determination of orbits of comets, minor planets, and satellites
Computation relating to air flight design

Analysis of circular shell-supported frames
Meteorological means
Earth tides
Evaporation computations
Boundary layer
Rolling pullout equations of motion
Raydist data analysis
Equations of pressure systems
Range error computation
Computations in connection with program analysis
Statistical smoothing rocket grain burning
Equations of combustion
Solution of sets of algebraic equations
Conversion of hexadecimal numbers
Airplane windshield deicing and defogging
Computations in connection with lattice arrangements
Tables of the bivariate normal distribution function
Pressure fields of potential flow past a body of revolution
Simplified rolling pullout equations
Frequency response study
Low moments of normal order statistics
Three non-linear differential equations
Probability distribution of Kolmogorov statistic
Reduction of hydrographic data

CHAPTER VI

THE PERIOD SUMMER 1951 THROUGH SPRING 1952

During Summer 1951 the research group was unusually large, in part, because a symposium was held late in August. Fritz John returned to New York University and Derrick Lehmer became the new Director. Rosser returned for the summer to participate in the program concerned with methods for solving linear equations and finding eigenvalues of matrices. Senior Research Personnel for the period July 1951 to June 1952 were:

Derrick H. Lehmer	(Director, UC Berkeley)
Forman S. Acton	(INA)
Shmuel Agmon	(Hebrew University, Summer 1951)
Lipman Bers	(NYU, Summer 1951)
Gertrude Blanch	(INA)
Leonard M. Blumenthal	(Missouri)
B. Vivian Bowden	(England, Fall 1951)
Alfred T. Brauer	(North Carolina, Summer 1951)
William E. Bull	(UCLA, Summer 1951)
Paul S. Dwyer	(Michigan, Summer 1951)
Gaetano Fichera	(Italy, Summer 1951)
Donald E. Fogelquist	(Sweden, Summer 1951)
George E. Forsythe	(INA)
Jerry W. Gaddum	(Missouri)
Herman H. Goldstine	(Institute for Advanced Study, Summer 1951)
Magnus R. Hestenes	(UCLA Liaison)
Gilbert A. Hunt	(Cornell, Fall 1951)
Harry D. Huskey	(INA)
William Karush	(Chicago, Summer 1951)
Tom Kilburn	(England, Fall 1951)
Cornelius Lanczos	(INA)
Theodore S. Motzkin	(INA)
Francis J. Murray	(Columbia)
Victor A. Oswald, Jr.	(UCLA)
Lowell J. Paige	(UCLA)
J. Barkley Rosser	(Cornell, Summer 1951)
David S. Saxon	(UCLA)
Isaac J. Schoenberg	(Pennsylvania)
Eduard L. Stiefel	(ETH, Switzerland)
Marvin L. Stein	(INA)
Philip Stein	(South Africa, Summer 1951)
J. G. van der Corput	(Holland, Summer 1951)
Joseph L. Walsh	(Harvard, Summer 1951)
Wolfgang R. Wasow	(INA)
Alexander Weinstein	(Maryland, Summer 1951)
Maurice V. Wilkes	(England, Fall 1951)

During Summer 1951 the special Junior Researchers and Graduate Fellows were:

Donald G. Aronson
Harold P. Edmundson
John H. Gay
Robert K. Golden
Robert M. Hayes
Kenneth E. Iverson
Thomas E. Kurtz
Richard H. Lawson
Theodore D. Schultz
William H. Warner
James P. Wesley
Mollie Z. Wirtschafer
Gloria Zinderman

During Fall 1951 through Spring 1952, the Junior Researchers were:

Charles E. Africa, Jr.
Richard H. Lawson
Daniel B. Ray
James A. Ward

and the Graduate Fellows were:

Richard G. Cornell
Stephen G. Gasiorowicz
Robert M. Hayes
Urs W. Hochstrasser
Michael J. Moracsik
Thomas Neill, Jr.
Anthony Ralston

Hochstrasser was a student from Switzerland who received his Ph. D. under the direction of Stiefel. In recent years Hochstrasser has been the Director of the Swiss Atomic Energy Commission. It is interesting to note that Iverson is now known as the originator of the programming language APL. Ralston is now a leading computer scientist.

Speakers at the INA Colloquium during Summer 1951 were:

I. J. Schoenberg
J. H. Curtiss
L. Bers
J. L. Walsh
J. B. Rosser
G. Fichera
E. Stiefel
F. J. Murray
J. G. van der Corput

During the period Fall 1951 to Spring 1952, we were fortunate in having the following visitors as colloquium speakers:

L. M. Blumenthal
Y. L. Luke
S. P. Frankel
I. J. Schoenberg
W. T. Reid
R. Isaacs
A. J. Hoffman

Derrick H. Lehmer was born in Berkeley, CA. His father was Professor of Mathematics at UC-Berkeley, specializing in Number Theory. After undergraduate studies at UC-Berkeley and graduate studies at Brown, he was a National Research Fellow at Caltech. After a short stay at Lehigh, he became Professor of Mathematics at UCB in 1940. One of the high points of his career was his selection by the American Mathematical Society to be the Gibbs Lecturer in 1965 [56]. In 1928 he married Emma Trotskaia. She, too, is a distinguished number theorist.

One of his early interests was in a mechanical number sieve operated photoelectrically. He later programmed sieves for general purpose machines and then built a special purpose electronic number sieve.

Lehmer had been involved in the dramatic investigation of the partition function by Hardy, Ramanujan, MacMahon, and later by Rademacher. Hardy and Ramanujan obtained an asymptotic formula for $p(n)$, the number of partitions of n. For $n = 200, p(n)$ is a number with 13 digits, which was computed by MacMahon. It turned out that 8 terms of the asymptotic series gave $p(n)$ with an error of 0.004. A refinement of the asymptotic formula used by Lehmer gave $p(721)$ as an integer (with 27 digits) $+0.0041$ suggesting strongly the exact value of $p(721)$. What Rademacher did was to obtain an identity for $p(n)$.

It was clear that sooner or later Lehmer would become the Director of INA. The "Oath Situation" at UC and the availability of the SWAC were enough to attract the Lehmers to INA.

Too few computers are designed for or by number theoreticians. The exploitation of the usual machines for number theory—even simple factorization routines—forces the user to develop programming tricks which are often of general value.

At INA Lehmer was concerned with the Ramanujan function $\tau(n)$, with the Riemann Hypothesis, and with Fermat's Last Theorem, the latter in collaboration with H. S. Vandiver. Earlier some work related to Fermat's Last Theorem had been carried out on the SEAC. Following a suggestion by H. Hasse, Olga Todd encouraged J. C. P. Miller to make experimental computations on consecutive pth power residues and on p itself as a pth power residue. This work was described in the New York Symposium [129]. Several later papers by the Lehmers are concerned with this problem.

As will be noted later some of the work on the SWAC was reported by Emma Lehmer at the Santa Monica Symposium [16].

Lehmer's selected papers have been published in three volumes [56].

An important addition to the Machine Development Unit was the appointment of Ragnar Thorenson. He was in charge of adding a drum to the SWAC, thereby significantly enlarging the memory of the SWAC.

Bull and Oswald were UCLA linguists who joined Harry Huskey in seeking methods for language translation by use of computers. They were pioneers in this field. Some of their results are given in [69]. There was also work on a Russian translation by Ida Rhodes of the Applied Mathematics Division in Washington.

Research in the Mathematical Theory of Program Planning was carried out enthusiastically by Motzkin, Agmon, Blumenthal, Gaddum, Schoenberg, and Walsh. During July and August a joint seminar was held with Rand on "Linear inequalities and related topics." Invited speakers from outside were: A. W. Tucker, R. W. Shepherd, J. M. Danskin, S. Karlin, and R. E. Bellman.

Studies in numerical integration of ordinary and partial differential equations were pursued vigorously by Agmon, Bers, Fichera, and Wasow. F. John completed Part I of an important memoir "On the integration of parabolic equations by difference methods" [40]. Johannes G. van der Corput continued his extensive research on methods of asymptotic expansions. Rosser investigated the problem of computing low moments of normal order statistics. He also participated in the program of finding effective methods for solving a system of linear equations.

Late in June or early in July, M. Hestenes devised a Conjugate Gradient Method for solving a system of linear equations. It is an n-step iterative routine. He gave three versions of this routine. When Stiefel arrived at INA from Switzerland, the librarian gave him a paper describing this routine. Shortly thereafter Stiefel came to Hestenes' office with this paper in hand and said, "Look! This is my talk." It turned out that Stiefel had invented the same algorithm from a different point of view. He looked upon it as a relaxation routine whereas Hestenes viewed it as a gradient routine on conjugate subspaces. The term "conjugate gradient" was coined by Hestenes. Because they had devised the same routine independently at about the same time, Stiefel and Hestenes decided to write a joint paper describing the routine and its properties. Hestenes arranged it so that Stiefel could remain at INA and UCLA for at least a 6-month period during which time the joint paper would be written (see [30]). During this period Stiefel taught a graduate course at UCLA on the Theory of Relativity.

Eduard L. Stiefel was a very versatile mathematician. He began his career as a topologist and made notable contributions in this field. One of these was used by Olga Todd in her proof of the impossibility of deriving the Laplace equation from the Cauchy-Riemann equations in dimensions other than 1, 2, 4, or 8. While at INA, Stiefel gave a proof of her result using representation theory.

In 1943 Stiefel was appointed to a professorship at ETH (Eidgenossischen Technischen Hochschule) in Zurich, Switzerland. He was well versed both in pure and in applied mathematics. He anticipated the coming of high-speed digital computers and was instrumental in the development of such a computer at ETH. This led him to an intensive study of numerical methods. His group at ETH became leaders in this field. They developed effective routines for solving large-scale systems of linear equations and for finding eigenvalues. Stiefel wrote an outstanding introductory book [96] on numerical analysis which was translated into several languages. Stiefel also made notable contributions to celestial mechanics (see, e.g., his book [97] with G. Scheifele). In addition, with the collaboration of his students, he wrote two other books [98,99]. Stiefel received many honors and participated in the city government of Zurich.

The conjugate gradient algorithm has, as a subroutine, an algorithm which is equivalent to an orthogonalization routine developed by Lanczos. The conjugate gradient routine therefore can be derived from the results given by Lanczos. Using his orthogonalization routine, Lanczos devised a "Method of Minimized Iterations" for solving a system of linear equations. This method is a variant of the original conjugate gradient routine. Credit should also be given to Rosser, Forsythe, Karush, and Paige for the development of the conjugate gradient routine, because the routine was also an outgrowth of their efforts. Rosser and Stiefel presented the conjugate gradient method at the August 23–25, 1951 Symposium on Simultaneous Linear Equations and the Determination of Eigenvalues, which was a part of the Semicentennial Celebration of NBS.

The technical organization of the symposium was mainly the responsibility of Olga Taussky-Todd, who, with assistance of L. J. Paige, edited the proceedings of the meeting. The proceedings were published by the National Bureau of Standards as Applied Mathematics Series 29. The symposium consisted of invited reports from various mathematicians, on selected topics, followed by a roundtable discussion. The reports on simultaneous linear equations were confined to finite systems, while those on the determination of eigenvalues dealt with both the discrete and continuous cases, and included reports specifically on the determination of bounds for eigenvalues. This symposium is considered by many to be the forerunner of the famous Gatlinburg Symposia. It was followed by one organized by Wallace Givens at Wayne State University, Detroit, in 1957 and then the Gatlinburg Series began, initially organized by A. S. Householder. They took place in 1961, 1963, 1966, and 1969. Later meetings took place at Los Alamos (NM) (1972), Bavaria (1976), Asilomar (CA) (1977), Oxford (England) (1981), and Waterloo (Canada) (1984). In his paper [35] Householder has written of ". . . the tremendous influence of INA in the early years of the development of modern numerical analysis. It would be hard to exaggerate this influence, and it would be harder to say, how much further along the subject might be now if the Institute had not been brutally cut off in its prime."

The papers presented at the 1951 conference are listed in Appendix D in the table of contents of AMS 29.

In this symposium the contributions made at INA were described by Forsythe, Hestenes, and Rosser. Forsythe presented a classification (and bibliography) of known methods for solving linear equations. Rosser described the contributions in this area by the staff at INA. Hestenes discussed gradient methods and Lanczos' method for finding eigenvalues and eigenvectors of matrices. The bibliography by Forsythe extended work begun by Ostrowski and the Todds. Others at INA, particularly Motzkin, collaborated in it. In a supplementary paper of the conference, Wallace Givens introduced his tridiagonalization method for finding eigenvalues and eigenvectors of symmetric matrices, a method which proved to be very useful and which became very popular.

There is a second set of important papers closely related to the papers described above, written under the sponsorship of NBS. These papers were published in NBS Applied Mathematics Series 39 in 1954, edited by Olga Taussky-Todd. The book is entitled "Contributions to the Solutions of Systems of Linear Equations and the Determination of Eigenvalues." Its contents are listed in Appendix D. A. I. and G. E. Forsythe described numerical experiments with accelerated gradient methods for solving linear equations. R. M. Hayes gave iterative methods for solving linear problems on a Hilbert space. J. Todd described the condition of finite segments of the Hilbert matrix.

This volume was followed by AMS 49, the title page and contents of which are listed in Appendix D. Two of the three papers in this volume were written by E. Stiefel and P. K. Henrici who were associated with INA.

As remarked above, Stiefel was given a joint appointment with INA and UCLA so that he and Hestenes could write an extensive exposition on the Conjugate Gradient Method for solving systems of linear equations. Several versions of the method were given. Numerical experiments were carried out by R. M. Hayes, Urs Hochstrasser, M. L. Stein, and W. Wilson. Wilson was a member of the computational staff. Unfavorable as well as favorable situations were considered. Even singular systems were studied. In each case the numerical results were consistent with theoretical ones when the effects of roundoff errors were taken into account. During this period Lanczos wrote up his version of this routine which he called a method of "Minimized Iterations." In addition he devised a special iteration for solving large systems of linear equations. This iteration was based on properties of Chebyshev polynomials. Variations of standard gradient methods with or without accelerations were studied experimentally and theoretically by Forsythe and M. L. Stein. Lehmer gave simple explicit expressions for the inverse, the characteristic polynomial, and any positive power of certain character matrices.

R. M. Hayes and M. L. Stein studied gradient and Rayleigh-Ritz methods for variational problems. In particular, Hayes was concerned with quadratic variational problems. Using techniques developed by Hestenes, he studied them in a Hilbert Space setting. Accordingly he developed a large class of iterative processes for solving linear self adjoint elliptic boundary value problems. One of these was a generalization of the conjugate gradient routine of Stiefel and Hestenes. In each case he established rates of convergence. His results were published in AMS 39 described earlier.

Forsythe initiated a study of Russian mathematical progress which led to the publication of a bibliography of Russian mathematics books [24]. Pertinent articles by Russians were collected and selected ones were translated into English by C. D. Benster under the editorships of Forsythe and Blanch. Some translations were published commercially [20,42]. Several appeared as NBS Reports [49,59,90]. An informal, but important, result of this program was the initiation of a class in Russian for mathematicians at UCLA and INA.

Studies in the mathematical theory of program planning and linear inequalities were continued vigorously by Motzkin, Schoenberg, Blumenthal, and Gaddum. A systematic treatment of the entire field of linear inequalities and their applications was undertaken in a graduate course by Motzkin at UCLA. Blumenthal continued his studies on metric methods in abstract algebra. Schoenberg pursued his theory of splines, a theory that has many useful applications.

Lehmer developed a practical method for obtaining the so-called Kloosterman Sums and investigated their properties. A series of tests for primality of Mersenne numbers were made on the SWAC, using a code sent in by R. M. Robinson of UC-Berkeley. Robinson wrote the Mersenne Code with a minimum of guidance. That the code was without error was (and still is) a remarkable feat. These tests showed a high degree of reliability in SWAC operation, and gave some unexpected information on the distribution of primes of the form 2^p-1. It was with a great deal of excitement that two new primes were discovered, namely, those for which $p=521$ and $p=607$. These were by far the largest known primes to this date. Later it was shown that $p=1279$ also yields a new larger prime. In addition, Lehmer and Golden made a numerical study of the periods in the continued fraction expansion of square roots of integers.

Studies in theoretical physics were carried out by Saxon in cooperation with members of the Physics Department and other departments at UCLA. The topics considered are illustrated by the following titles of papers that they produced: "Modes of vibration of a suspended chain"; "Distribution of electrical conduction currents in the vicinity of thunder storms"; "Radiation characteristics of a turnstile antenna shielded by a metallic tube at one end"; "An optical model for nucleon-nuclei scattering"; "Variational calculation of scattering cross-section"; and many others. It is interesting to note that the paper, "Modes of vibration of a suspended chain," answered a query by E. U. Condon, the Director of the National Bureau of Standards, who had noted that the traditional solution required the end supports to restrain infinite pulling forces.

CHAPTER VII

THE PERIOD SUMMER 1952 THROUGH SPRING 1953

As usual there was increased activity in research during the summer period. Senior Research Personnel and seminar participants for the period Summer 1953 through Spring 1953 were:

Derrick H. Lehmer	(Director, UC Berkeley)
Forman S. Acton	(INA)
Adrian A. Albert	(Chicago, Summer 1952)
Edward W. Barankin	(Berkeley)
Gertrude Blanch	(INA)
Leonard M. Blumenthal	(Missouri, Summer 1952)
Truman A. Botts	(Virginia, Summer 1952)
Johannes G. van der Corput	(Holland, Summer 1952)
George E. Forsythe	(INA)
Jerry W. Gaddum	(Missouri, Summer 1952)
Dick W. Hall	(Maryland)
E. H. Hanson	(Texas State, Summer 1952)
G. A. Hedlund	(Yale, Summer 1952)
Magnus R. Hestenes	(UCLA Liaison)
Gilbert A. Hunt	(Cornell, Summer 1952)
Harry D. Huskey	(INA)
Mark Kac	(Cornell, Summer 1952)
Irving Kaplansky	(Chicago, Summer 1952)
William Karush	(Chicago, Summer 1952)
Cornelius Lanczos	(INA)
Richard A. Leibler	(Washington, DC, Summer 1952)
Harold Levine	(Harvard, Summer 1952)
Arvid T. Lonseth	(Oregon State, Summer 1952)
Theodore S. Motzkin	(INA)
Victor A. Oswald Jr.	(UCLA, Summer 1952)
Lowell J. Paige	(UCLA)
David S. Saxon	(UCLA)
Isaac J. Schoenberg	(Pennsylvania, Summer 1952)
Julian Schwinger	(Harvard, Summer 1952)
Dan Teichroew	(INA)
Charles B. Tompkins	(INA)
Joseph L. Walsh	(Harvard)
James A. Ward	(Utah, Summer 1952)
Wolfgang R. Wasow	(INA)
Charles Wexler	(Arizona State, Summer 1952)
J. Wolfowitz	(Cornell)

Junior Researchers and Graduate Fellows were:

John W. Addison, Jr.	Charles E. Africa Jr.
Fred Baskin	Richard G. Cornell
Walter I. Futterman	Stephen G. Gasiorowicz

Robert M. Hayes	Urs W. Hochstrasser
Richard A. Lawson	Stanley W. Mayer
Michael J. Moravcsik	Thomas Neill
Anthony Ralston	Daniel B. Ray
John Selfridge	Marion I. Walter
Roger D. Woods	

Lectures in the Numerical Analysis Colloquium Series were given by the following visitors:

A. M. Ostrowski	A. T. Lonseth
G. A. Hunt	M. Kac
D. Ray	H. Levine
J. G. van der Corput	G. Pólya
E. W. Barankin	P. Erdös
A. Dvoretzky	J. Wolfowitz
M. M. Schiffer	M. Minorsky
R. De Vogelaere	S. Bochner

Robert Hayes received his Ph. D. upon writing a thesis under the direction of M. R. Hestenes. His thesis was concerned with iterative methods for solving linear problems in Hilbert Space [27]. During his stay at INA he became interested in information storage and retrieval. This eventually led him to the use of computers in libraries. He became a pioneer in this field and is now the Dean of the Library School at UCLA.

During Summer 1952 a Seminar on Finite Projective Planes was held under the supervision of C. B. Tompkins. The principal participants in this seminar were Albert, Botts, Hall, Hanson, Hedlund, Hunt, Kaplansky, Leibler, Paige, Tompkins, Ward, and Wexler. Lectures were given by C. B. Tompkins, A. M. Gleason, J. A. Ward, L. K. Frazier, D. W. Hall, L. J. Paige, D. H. Lehmer, A. A. Albert, and M. Hall, Jr. This seminar was headed by S. S. Cairns of the University of Illinois. It was a forerunner of the SCAMP Project which was set up during the following year. Related work was carried out jointly by A. J. Hoffman, M. Newman, E. Straus, and Olga Taussky-Todd. Olga Taussky-Todd discussed topics in this area in her 1952 lecture in Palermo (see J. Research NBS 65 (1961), 15–17) using work of G. Pall and M. Newman.

Lehmer introduced a new project designed to investigate the logical theory of digital computing machines, particularly of automatic type, and to discover new techniques for using such machines. The projects introduced earlier were continued in the usual manner. Barankin joined Motzkin in his project on linear inequalities. Schwinger gave a series of lectures on quantum dynamics. Lanczos devised four routines for using Chebyshev polynomials in solving large-scale systems of linear equations. Lonseth continued his research on the numerical solution of integral equations by an adaptation of least square techniques. Southard and Yowell devised an alternative "predictor-corrector" process for the numerical integration of ordinary differential equations with initial conditions. A new Mersenne prime was found with $p=2203$. Numerical experimentation on solving linear equations and finding eigenvalues was continued on the SWAC.

Acton, Huskey, and Lanczos were on leave of absence during fiscal year 1953. During his leave of absence, Huskey was employed by Wayne University for the purpose of setting up a computation laboratory. As usual, Hestenes and Paige participated on a part-time basis.

C. B. Tompkins became a regular member of the INA staff. He was a very versatile mathematician who was well versed in applications of mathematics and in the engineering aspects of computing machines. He received his M.A. degree at the University of Maryland in chemistry, physics, and mathematics. He took his Ph. D. at the University of Michigan in 1935. After a year as an instructor at Maryland, he spent 2 years as a National Research Fellow at Princeton University and the Institute for Advanced Study, followed by 3 years as instructor at Princeton. During these years Tompkins collaborated with Marston Morse in his work on the Calculus of Variations in the Large. He also held a Visiting Lectureship at the University of Wisconsin. During World War II Tompkins served as a naval communications officer in the Pacific, rising to the rank of Lieutenant Commander. His experiences there impressed upon him the need for the development of high-speed computers. In 1946, he became one of the founders of Engineering Research Associates in St. Paul, MN, a firm which played an important role in the development of the United States computer industry. In 1949 he organized, on behalf of the Office of Naval Research, the Logistics Research Project at George Washington University. He served as consultant and member of advisory panels to various governmental agencies. In this connection, he played a major role in the development Project SCAMP which we refer to later. This project later became an integral part of the Institute for Defense Analyses in Princeton, NJ. He also served as a consultant to various computer corporations. As noted earlier, Tompkins served as consultant to Mina Rees and John Curtiss in the establishment of the National Applied Mathematics Laboratories at NBS, of which INA was an integral part.

In Spring 1953, Tompkins instituted a Seminar on Numerical and Computational Aspects of Linear problems: Games, Linear Equalities, Linear Inequalities, Programming, etc. In this seminar R. B. Horgan described SWAC coding of a search for a solution to a large set (128) of simple linear inequalities in many (80) variables. M. Weber and E. C. Yowell described their SWAC experience on finding the eigenvalues of a 45×45 matrix. Horgan, Weber, and Yowell were members of the Mathematical Services Unit at INA with Yowell in charge of machine computations.

Because Harry Huskey was on a leave of absence during fiscal year 1953, Ragnar Thorensen was in charge of the Machine Engineering and Development Unit. By the end of the year SWAC was completed with the successful installation of a magnetic drum. Thorensen also redesigned the Williams tube circuitry of the SWAC, thereby improving its reliability. The SWAC was now working with greater efficiency. This increased the performance capabilities of the Mathematical Services Unit and enabled the research staff to experiment with "larger" problems.

Studies in matrix inversion and in matrix eigenvalue problems were continued. Numerical integration of differential equations was investigated further. Work in the mathematical theory of program planning continued at approximately the same level. An investigation of variational methods applied to quantum mechanical scattering problems gave valuable results. Translation of interesting Russian mathematical articles was continued. The SWAC was used on a low priority basis to investigate interesting topics in pure mathematics. A new Mersenne prime with $p = 2281$ was discovered.

Several disturbing incidents occurred at INA during this period. One was the suspension of an administrative clerk by the Commerce Department Loyalty Board No. 2, an action that appeared to be unjustified by the evidence as presented in the abstract of the hearing. Other personnel were similarly investigated but cleared. These actions had an extremely negative effect upon the morale of the entire institution. Equally disturbing was the unsettling relationship of the National Bureau of Standards with other governmental agencies as a result of the "Battery Additive" case. By the end of this period an "austerity" program had to be initiated at INA.

It is appropriate to digress for a moment and describe the situation that arose at NBS which led to the austerity program at INA and finally to its separation from NBS. The NBS Applied Mathematics Division (AMD) was chronically in a precarious financial status in that it depended largely on transfer of funds. In the late forties money was plentiful and AMD flourished. An important factor was the central location of NBS in Washington, DC. It was easy for John Curtiss and the senior members of his staff to keep in contact with the mathematical establishment as its members visited Washington on committee business. These meetings generated goodwill toward AMD, and in particular, simplified recruiting. (The present suburban location makes casual contacts less easy.)

To indicate the extent of the dependence of the AMD on transfer of funds, we note that, in Fiscal Year 1953, out of its total budget of $1,546,200, only $128,400 came from NBS appropriations. In that year there were 163 employees in the Division.

During these, on the whole, good times there were occasional singularities. We mention some.

It would have been too much to expect that the staff could all avoid the security investigations of the McCarthy era. John Curtiss saw to it that any of his staff involved got the best advice. One of us (J. T.) recalls calling with Curtiss on a tennis friend to get advice from his father, who was an important figure in Washington legal circles, on how to proceed.

The continued harassment of Condon contributed to his decision to resign and join the Corning Glass Company as Director of Research in 1951. He later returned to academic life at Washington University, St. Louis, and then at the University of Colorado, where he died in 1974. Condon was succeeded as Director of NBS by Allen V. Astin, who was very sympathetic to the AMD, though not quite as close as Condon.

Another great blow to the AMD was the resignation of John Curtiss in mid-1953. In Appendix A we reproduce an evaluation of his work by John Todd [115]. F. L. Alt, who had been with the AMD in various capacities, took over as Acting Chief and deserves great credit for holding the organization together in very distressing circumstances until E. W. Cannon was appointed Chief in late 1954. He remained in this position until his retirement in 1972. Cannon had been Assistant Chief of the AMD in various capacities since its formation, and was on leave of absence during 1953–54 to serve as head of the George Washington University Logistic Research Project.

Another conflict was with the business sector. The NBS was supposed not to be in competition with private enterprise. However, on occasion a Government contractor, who needed computational help which NBS alone could give efficiently at that time, would arrange for funds to be diverted from his contract, say with the Department of Defense, to the NBS. There were a few complaints that the INA, using Government supplied equipment, was undercutting the small business computer firms which were then springing up.

However, the critical blows came from elsewhere. It was a consequence of the AD-X2 (battery additive) controversy in which the NBS was involved from 1948 to 1953.

Part of the statutory responsibility of the NBS from the beginning was to render advisory services to Government agencies on scientific and technical problems. Thus from the early 1920's the NBS had tested a great many commercial battery additives, principally for the Post Office in connection with mail fraud and for the Federal Trade Commission (FTC) with reference to public advertising. All were of much the same composition, and distributed with more or less the same claims. The NBS found none to be beneficial. *NBS Circular 504*, issued in 1951 after a new series of such tests, warned people to be suspicious of all such products.

The proprietor of a small California company proclaimed that his product, the AD-X2, had a special composition and beneficial effect. The company encouraged satisfied users to flood Congress and the NBS with testimonial letters and, by the end of 1951, had gained the support of 28 Senators and 1 Congressman. Early in 1952, the NBS performed further tests of AD-X2 at the request of the FTC, the Post Office, and the Small Business Committees of the House and Senate. No beneficial effect of AD-X2 treatment was found. In the hope of finally resolving the issue, the NBS performed a "blind" comparison test of matched groups of treated and untreated batteries approved by the manufacturer following a procedure mutually agreeable to him and the NBS. The comparison test, carried out carefully by staff of the Electrochemistry Section of the Electricity Division, with meticulous analysis of the resulting measurements by the Statistical Engineering Laboratory of the AMD showed that the product had no beneficial effect.

About the same time laboratory tests of this product were made by a group at a major university. The group's report did not claim that any practical benefits were associated with the observed effects but the Senate Select Committee on Small Business was informed by a consultant thereto that these tests completely supported the claims of the manufacturer, and told this to the world by press release in December 1952.

In January 1953 Eisenhower succeeded Truman as President. Sinclair Weeks became Secretary of Commerce and Charles E. Wilson became Secretary of Defense. On March 24, 1953, Dr. Astin was asked to resign by the Assistant Secretary of Commerce for Domestic Affairs to whom NBS reported. Weeks explained ". . . I think that the National Bureau of Standards has not been sufficiently objective, because they discount entirely the play of the market place . . ." [Senate Small Business Committee hearing on AD-X2, March 31, 1953; *Hearings* vol. p. 3; *Washington Post*, April 1, 1953, p. 1]. Astin's resignation was submitted on March 30, and was accepted by President Eisenhower on April 2, to be effective April 18.

It is easy to imagine what alarm and despondency the news of Dr. Astin's forced resignation brought to the NBS community. The first public announcement was probably in Drew Pearson's column in the *Washington Post* of March 31. Among the actions considered was a mass resignation in sympathy with Dr. Astin [*Evening Star*, April 17, p. 1]. The NBS community was relieved by the outbursts in the press which included incisive cartoons. The nation's scientific community erupted in overwhelming support of Dr. Astin and the integrity of the NBS.

In response to a suggestion of Sinclair Weeks, Secretary of Defense Wilson issued an order on April 4 stating, among other things, that "Effectively immediately, no funds shall be obligated by any Department or Agency of the Department of Defense for research and development to be performed by a Government agency outside the Department of Defense without specific approval of the Secretary or Deputy Secretary of Defense . . ." [Date of order given and order quoted in *Washington Post*, April 24, 1953, p. 8]. This caused considerable anxiety at INA, which was supported almost entirely by the Office of Naval Research and the Air Force.

Late on Friday, April 17, 1953, the day before Astin's resignation was to take effect, Weeks announced (1) that the National Academy of Sciences would establish two committees; one (the Jeffries Committee) to appraise the quality of the Bureau's work in relation to AD-X2 and to determine whether additional testing was needed, and another (the Kelly Committee) to evaluate the functions and operations of the NBS in relation to the nation's needs; and (2) that he had asked Dr. Astin, and Dr. Astin had agreed, to continue to serve at least until the Academy Committees had completed their work, but "no question is involved of Dr. Astin's permanent retention."

Although no mathematician was on the Kelly Committee initially, J. Barkley Rosser was appointed to it subsequently. Mathematical statisticians, W. G. Cochran and S. S. Wilks, served on the Jeffries Committee.

It happened that Astin had been invited to address the American Physical Society on May 1, 1953. In this memorable address he enunciated the beliefs of the Director of NBS and its staff, beginning as follows:

> "The Bureau staff believes first of all in the importance of scientific research as a means of intellectual and scientific advancement, as the foundation of our technological economy and high standard of living and as the bulwark of our national security.
>
> We believe in the teachings of Galileo that theory and hypothesis must conform to the results of experimentation and observation. We believe in the philosophy of Lord Kelvin, that basic understanding in science depends on measurement—the reductions of observation to numbers."

Astin's address was presented to an overflow audience which gave the speaker a standing ovation., We recommend the study of the whole address. It appeared in *Physics Today* 6 (1953), #6, 12–13 and is reprinted in the Astin Memorial Symposium booklet (NBS Special Publication 690, January 1985).

The Kelly Committee submitted its initial findings late in July 1953, and its formal report on October 15, 1953. Its appraisal of the functions and operations of the NBS was in the main favorable. Its principal recommendation was that the Bureau's weaponry development program be transferred to the Department of Defense.

On August 21, a Friday afternoon, Dr. Astin was reinstated. One of us (J. T.) recalls rumors about the decision on that day and being depressed when there was no news at the close of business. However, we remained in our office until Churchill Eisenhart later brought us the good news which had been held up to prevent any demonstrations. The NBS was put under the supervision of the Assistant Secretary of Commerce for Administration.

On September 27, 1953, four major divisions of the NBS with about 60 percent of the personnel of the NBS were transferred to the Department of Defense. The three working on proximity fuzes and related materials became the Diamond Ordnance Fuze Laboratories of the Army Ordnance Corps. The Bureau's Missile Development Division at Corona, CA, became the Naval Ordnance Laboratories (Corona).

Summaries of the Kelly Committee report are given in *Science*, 119 (1954), 195–200, and in *Physics Today* 6 (1953) #11, 17, #12, 4–11. While the Kelly Report highly praised the AMD activities, it questioned the appropriateness of the INA as part of the NBS program but left the decision on the future of INA to the AMAC. After much deliberation it was decided that the NBS had to give up the sponsorship of INA on June 30, 1954. The dissolution of INA is described in the next chapter.

The Jeffries Committee fully supported the decisions of NBS on AD-X2. Summaries of its report are given in *Science,* 118 (1953), 683-5 and *Physics Today* 6 (1953) #6, 12 and #12, 26. There is also a detailed account in [53] Chapter 3, 16–60, "AD-X2: The difficulty in proving a negative." Another account is given by Samuel A. Lawrence [54]. A non-technical account of "The AD-X2 affair" is given on pages 12–21 of the Allen V. Astin Symposium booklet [71].

CHAPTER VIII

THE PERIOD SUMMER 1953 THROUGH SPRING 1954

D. H. Lehmer returned to the University of California at Berkeley in August and C. B. Tompkins took over the Directorship of INA. Senior Research Personnel, including participants of the SCAMP Project were:

Derrick H. Lehmer	(Director, UC Berkeley)
Charles B. Tompkins	(Director, UCLA)
Gertrude Blanch	(INA)
Harry D. Huskey	(On leave)
Forman S. Acton	(On leave)
Alfredo Baños Jr.	(UCLA, Summer 1953)
Richard H. Bruck	(Wisconsin, Summer 1953)
Kenneth A. Bush	(Illinois, Summer 1953)
Randolph Church	(Monterey, Summer 1953)
Robert P. Dilworth	(Caltech, Summer 1953)
George E. Forsythe	(INA)
Richard A. Good	(Maryland, Summer 1953)
John W. Green	(UCLA, Summer 1953)
Dick Wick Hall	(Maryland, Summer 1953)
Marshall Hall Jr.	(Ohio State, Summer 1953)
Magnus R. Hestenes	(UCLA Liaison)
Erwin Kleinfeld	(Ohio State, Summer 1953)
Harold W. Kuhn	(Bryn Mawr, Summer 1953)
Cornelius Lanczos	(On leave)
Jacob Marschak	(Chicago, Summer 1953)
Theodore S. Motzkin	(INA)
Lowell J. Paige	(UCLA, Summer 1953)
Gordon Pall	(Illinois Tech., Summer 1953)
William A. Pierce	(Syracuse, Summer 1953)
David S. Saxon	(UCLA, Summer 1953)
Jonathan D. Swift	(UCLA, Summer 1953)
Daniel Teichroew	(INA)
Joseph L. Walsh	(Harvard, Summer 1953)
Wolfgang R. Wasow	(INA)

Acton, Huskey, and Lanczos remained on leave. S. S. Cairns was in charge of Project SCAMP, a classified project sponsored by the Department of Defense. It was generally believed that it was concerned with problems of communications.

The Graduate Fellows were:

John W. Addison Jr.	Walter I. Futterman
Eugene Levin	Genovevo C. Lopez
Stanley W. Mayer	Edwin Mookini
Mervin Muller	Lloyd Philipson
John Selfridge	Marion I. Walter
Roger D. Woods	

Most of the Graduate Fellows and many of the members of the Computing Staff obtained Ph. D.'s in their field of specialization. Most of these hold university positions. Others are in industry and in government service. They are scattered throughout the United States and elsewhere. Their contributions have been significant. In the present group, for example, John Addison is Professor of Mathematics at UC-Berkeley. Edwin Mookini (now deceased) became Vice President of the University of Hawaii and was the top candidate for the Presidency at the time of his death. Mervin Muller, now at Ohio State was in charge of computing at the World Bank. John Selfridge has held professorships at various universities and was a very successful Managing Editor of *Mathematical Reviews* during the period of its computerization.

The speakers at the Numerical Analysis Colloquium Series were:

G. Pall	S. Lefschetz
J. Marschak	R. Radner
A. Baños Jr.	F. G. Foster

The National Bureau of Standards was a co-sponsor with the American Mathematical Society of a Symposium on Numerical Analysis held at Santa Monica City College, August 26–28. John H. Curtiss was the chairman of the organizing committee. The symposium was entitled "American Mathematical Society Sixth Symposium in Applied Mathematics: Numerical Analysis." The proceedings were published in 1956 with John Curtiss as editor. The title page and table of contents is given in Appendix D.

The papers presented in this symposium were concerned mainly with the following topics: discrete variable problems, combinatorial problems, number theory, eigenvalues, solutions of linear systems, partial differential equations, linear and dynamic programming, and approximation theory. A large number of the participants had been associated with NBS and INA as visiting scientists. NBS and INA were represented by C. B. Tompkins, Olga Taussky-Todd, Emma Lehmer, M. R. Hestenes, T. S. Motzkin, and W. R. Wasow. M. R. Hestenes presented a generalized conjugate gradient routine for solving linear systems. It was shown that every n-step procedure can be interpreted to be a conjugate gradient routine of this type.

During Summer 1953 research on projects introduced earlier was continued. A special n-step process for solving linear systems was devised by Motzkin. Walsh joined Motzkin in the study of approximating polynomials. Studies in linear programming were continued by Motzkin, Kuhn, and Marschak. Tompkins was concerned mainly with discrete variable problems. He was joined by Kleinfeld and the Lehmers. Of course, number theory was pursued by various members of the group.

David Saxon returned to his position in the Department of Physics at UCLA. He had a distinguished career in research and in administration. In 1975 he became President of the University of California. In 1983 he was appointed Chairman of the Corporation of Massachusetts Institute of Technology.

We now begin the final period of existence of INA. Research went on as usual. Numerical experiments were made in finding the inverse of a matrix A. This led to the following problem. Given an estimate X of the inverse of a matrix A, how do you improve this estimate? The Newton algorithm of replacing X by $(2I-XA)X$ was tried. In this algorithm the calculations of the matrices $E=2I-XA$ and EX were crucial. When high (triple) precision was used, the algorithm was highly successful for the matrices studied. When single precision was used throughout, the estimates became worse and worse. This phenomenon has been investigated deeply by J. H. Wilkinson.

Methods of preconditioning a symmetric matrix A by pre- and post-multiplication of A by a diagonal matrix D were studied in [23] by Forsythe in cooperation with E. G. Straus of UCLA. Precise characterizations of "best conditioned" matrices were obtained. Preconditioning tends to reduce roundoff errors in matrix computations.

Studies in numerical solutions of certain partial differential equations were carried out by Blanch, Motzkin, Philipson, Tompkins, and Wasow. Incidentally, Philipson wrote his Ph. D. dissertation under the supervision of Wasow.

The Seminar in Numerical Analysis attracted many visitors from local industries and from UCLA and nearby universities. It was held twice a week and was devoted to applications of numerical analysis to mathematics, physics, engineering, and economics.

The decision of Secretary of Defense, Charles E. Wilson, to no longer permit a non-DOD Government agency to serve as administrator of projects carried out at a university but supported entirely, or in large part, by DOD funds, caused the National Bureau of Standards to give up its administration of INA by June 30, 1954. The University of California was invited to take over this administration. The university was not in a position to take over all sections of INA. However, UCLA agreed to take over the administration of the research group, the SWAC and its maintenance, and the Library. This responsibility was assumed under contracts with the National Bureau of Standards, the Office of Naval Research, and the Office of Ordnance Research. The contract with NBS provided for the loan of the equipment which had been at the Institute. The question of faculty status of INA members was to be dealt with after the takeover had been accomplished. The new organization was to be called Numerical Analysis Research (NAR). The local negotiation team was comprised of C. B. Tompkins, M. R. Hestenes, Dean Paul Dodd, and Dean Vern Knutson.

At this time, a Numerical Analysis section was set up at NBS in Washington with John Todd as Chief and with, on a smaller scale, a mission similar to that of INA. Milton Abramowitz became the Chief of the Computation Laboratory, much reduced in size because of the restriction on transferred funds. An account of "Numerical Analysis at the NBS" in the first 25 years 1946–71 was given by John Todd in *SIAM Review* [116]. In 1957 John and Olga Todd joined the Department of Mathematics at California Institute of Technology.

From the beginning of INA, John Curtiss worked diligently to build a first class research library at INA. He was ably assisted by Forsythe with the help of Motzkin and Tompkins. They were very successful. When INA was transferred to UCLA, the Department of Mathematics became the custodian of the Library. Tompkins and Hestenes were successful in obtaining funds for maintenance and further development of the Library. Due to the efforts of L. J. Paige, the Library was eventually transferred to UCLA. This gift enabled the Department of Mathematics to build a first class Departmental Library. In appreciation of this gift, the books and journals received are listed as belonging to the NAR Collection.

John Curtiss, working in conjunction with the various directors of INA, always encouraged the departments of UCLA and neighboring universities to make use of the computing facilities at INA. Accordingly, there were a large number of unofficial members of INA comprised of faculty and students who carried out pioneering research in various fields. Notable among these were Michel Melkanoff of the Physics Department; James McCullough, Robert Sparks, and Kenneth Trueblood of the Chemistry Department; Andrew Comrey of the Psychology Department; Yale Mintz of the Meteorology Department; and Russell O'Neill of the Department of Engineering. According to David Saxon "The work that was done on the optical model in those early days was a genuine departure on the theoretical side of physics because it did permit the attack on an unusually complicated problem which would have been otherwise impossible, and for some time the program which was written first by Roger Woods and then by Melkanoff was the most complicated computational program of any that we knew about in the world of theoretical physics." This program, developed under the leadership of David Saxon, became a world-wide standard for analyzing experiments on scattering of all kinds of particles (protons, neutrons, mesons, alphas) and became a key tool for experimental physicists. Sparks and Trueblood wrote, "Digital computers have revolutionized the practice of crystallography in the last thirty years. . . .One of the greatest triumphs of early computers was SWAC's contribution to the solution of the hexacarboxylic acid derivative of the vitamin B12."

INA attracted many distinguished visitors such as, John von Neumann, Solomon Lefschetz, Edward Teller, Norbert Wiener, and many others, including researchers from neighboring universities. These were in addition to the many visitors who were given temporary appointments at INA. This made INA an exciting place to be, not only for the regular members of INA, but also for the visitors and the unofficial members of INA.

The engineers resigned on November 1, 1953 and accepted positions with the Magnavox Corporation. The responsibility for maintaining the SWAC was assumed eventually by Fred H. Hollander.

By June 30, 1954, various members of INA had accepted positions in industry and in various departments of universities. For example, B. Handy, A. D. Hestenes, M. Howard, and E. C. Yowell were employed by National Cash Register. S. Marks and A. Rosenthal went to the Rand Corporation. These and others continued to make significant contributions in applications of computers.

During his leave of absence from INA, Lanczos was employed by North American Aviation as a specialist in computing. In 1954 at the invitation of Eamon de Valera, who was at that time Prime Minister of the Republic of Ireland, Lanczos accepted the post of Senior Professor in the School of Theoretical Physics of the Dublin Institute for Advanced Studies. There he had stimulating discussions with not only the senior staff but also with the junior staff of the Institute. He published many scientific papers and six standard works written in a very personal style. These standard works are: *Applied Analysis*, 1956; *Linear Differential Operators*, 1961; *Albert Einstein* and the *Cosmic World Order*, 1965; *Discourse on Fourier Series*, 1966; *Numbers Without End*, 1968; and *Space Through the Ages*, 1970. His last book, *Einstein Decade: 1905–1915*, was published in 1974. The number of his scientific papers was about one hundred. Lanczos received many honors, such as, Membership of the Royal Irish Academy, 1957; the award of the honorary D. Sc. by Trinity College, Dublin in 1962; the honorary degree of D. Sc. by the National University of Ireland in 1970; the honorary degree of Dr. Nat. Phil. from the Johann Wolfgang Goethe University, Frankfort, in 1972; and the honorary degree of D. Sc. from the University of Lancaster in 1972. Although his work in his later years was concerned mainly with mathematical physics, he kept his interest in numerical analysis and his influence on researchers in this field was great. Lanczos was awarded the Chauvenet Prize of the Mathematical Association of America for his 1958 *Monthly* paper "Linear systems in self adjoint form." He died in 1974.

In 1954 Harry Huskey accepted a faculty position at UC-Berkley, where he continued to make significant contributions in the computer field. In 1967 he moved to UC-Santa Cruz to serve as Professor of Computer and Information Science. There he set up the USCS Computer Center and served as its Director from 1967–1977. Internationally, he was in great demand as a consultant to various computer centers, e.g., centers in India, Pakistan, Burma, Brazil, and Jordan. He spent an extended period in Kanpur, India, where he directed the setting up of a computer center. Huskey has served as a consultant to industrial and governmental agencies. He has been very active in professional organizations on computers. He has received many honors. In particular, he and the members of the SWAC Staff were honored at the Pioneer Session of the National Computer Conference held in Anaheim, CA for their early work on the SWAC.

George E. Forsythe was one of the senior members of INA who remained with NAR. He soon was given a faculty appointment in the Department of Mathematics, where he was in charge of the educational program in numerical analysis. At his suggestion, Peter Henrici of Zurich, who had been earlier attached to NBS and INA, was invited to join the Department. Henrici accepted, thereby strengthening the program in numerical analysis at UCLA. Henrici was an expert in numerical solutions of differential equations by finite difference methods, as well as an authority on the theory of functions of a complex variable. In 1957 Forsythe received a very attractive offer of a faculty position at Stanford University, which would enable him to set up a program of his own. He accepted this offer. In 1961 he became Professor of Computer Science and Chairman of the Department of Computer Science. Under his leadership, this department became the most influential one in the country, attracting almost as many National Science Foundation Fellows as all other such departments combined. Forsythe was for 2 years a member of the Board of Trustees of SIAM. He, as well as Huskey, served a term as President of the Association of Computing Machines (ACM), and has been otherwise active in these and other professional organizations. His wife, Sandra (Alexandra), was for some time a member of the staff at INA and assisted him in various significant computational programs. Later, at Palo Alto, she made significant contributions to the teaching in computer science at the high school level. Forsythe died in 1972. A tribute to him is given by A. S. Householder in the *SIAM Journal on Numerical Analysis* [36]. See also the article by J. Varah in [131].

Wolfgang Wasow was also a senior member of INA who remained with NAR. However, he held Fulbright Scholarships in Rome, Italy (1954–55), and in Haifa, Israel (1962). He spent the academic year 1956–57 at the Army Mathematics Research Center in Madison, WI. In 1957 Wasow joined the Department of Mathematics at the University of Wisconsin, where he has had a distinguished career. In 1960 Forsythe and Wasow published a very successful book entitled *Finite-Difference Methods for Partial Differential Equations.* It was translated into Russian, Japanese, and Chinese.

Theodore S. Motzkin became a member of the Department of Mathematics at UCLA. He attained an international reputation for his contributions in the fields of linear inequalities, combinatorics, convexity, and approximation theory. He wrote significant papers in various fields, such as, algebraic geometry, algebra, number theory, function theory, and graph theory. He was a stimulus to his colleagues. Motzkin was sought as visiting professor, as organizer and principal speaker of many institutes, including two on combinatorics and inequalities, and as editor of journals and symposia. Motzkin died in 1970.

Charles B. Tompkins became a member of the Department of Mathematics at UCLA. He was in charge of the NAR Project and continued to be active in the SCAMP Project. He continued to make the computing facility available to all interested faculty and students. In 1961 the Computing Facility was set up as a separate organization with Tompkins as its first Director. Michel Melkanoff played a significant role in the setting up of this organization. Tompkins was continually advocating the broadening of the mathematical curriculum and organizing seminars. He was convinced that mathematics and computers would have an increasing role to play in many new fields of research, and, accordingly, he enthusiastically organized and conducted interdisciplinary colloquia. Tompkins died in 1971. A short summary of his career was published in *Mathematics of Computation* [122]. The paper [121] is representative of one area of his work.

Thomas H. Southard also remained with NAR. He became a member of UCLA-Extension and contributed significantly to its applied mathematics program with special emphasis on numerical analysis. Later he accepted the Chairmanship of the Department of Mathematics at the California State College at Hayward, where he continued to emphasize the need for a broad mathematical curriculum.

By and large the policies of INA became policies of NAR. The research program was continued much in the same manner as before with the support of ONR. Visitors continued to be invited and to be supported by NAR. Qualified graduate students were supported. The SWAC remained available for research computations for interested departments at UCLA and neighboring institutions. The SWAC was officially retired in 1967. In the meantime a separate computing facility at UCLA was set up to serve the entire university with the help of a more modern IBM electronic computer, a gift from IBM. Earlier, the Business School, in cooperation with IBM, set up the Western Data Processing Center to serve the business schools of the west coast universities. Later, a Computer Science Department was organized by the Engineering Department. Many members of this Department had early training on the SWAC. A separate computing facility supported by the National Institute of Health was set up in the UCLA medical school for medical research. Tompkins and Hestenes played a significant role in the establishment of these computing facilities at UCLA. The Institute for Numerical Analysis therefore enabled UCLA to be a pioneer in the use of computers for research. In addition graduates of INA became leaders in numerical research elsewhere.

At 12 o'clock on Wednesday, June 30, 1954, INA held a "Gala Party and Luncheon" in honor of the transfer of INA to UCLA, the people leaving INA, and the persons staying in NAR. At least 77 persons were present including David Saxon, the recent President of the University of California. Shirley Marks, a long-time employee of INA, composed and read the following poem:

Looking back may be a sign of age
But, happily, may also be a gauge
Of what's to come, of future's promise—
And so reassure the doubting Thomas.

So let's return to the year '48
And see what, together, we can recreate
Of the life and times of INA—
Before they, old-soldierly, fade away.

Remember the summer of B. A. C.?
Before Air-Conditioning: and the DDT
On desks and clothes; and Rusty the Rabbit
Whose tribe increased, as if by habit?

And the ping-pong tables we never made,
And the welfare meetings forever delayed
By erudite speeches on cookies and cokes,
And the cakes Mrs. Milne baked, and shaggy-dog jokes?

The visitors in summer of world-wide renown
Who politely inquired if the SWAC was still down?
And the picnics, and parties, and two hours for lunch,
And Rudolph the Reindeer, and lime-sherbet punch?

Dr. Lanczos exhibiting his thespian powers,
The Lehmers code-checking into wee hours,
SWAC in the movies, Scrabble, Hearts, good friends?
On this happy past INA's future depends.

And though it begins with a change of name,
(Appropriate in June), the face is the same.
And so, all good wishes to those who depart,
And to those who remain—best of luck—from the heart.

With this poem the saga of INA ended and NAR was born.

CHAPTER IX

SOME TYPICAL PROJECTS

In this chapter we shall discuss, with a little more technical detail, some typical projects that were carried out at INA. This will indicate the flavor of the work done at INA. We have chosen projects in which we have had a special interest. These projects were carried out, in part, with the cooperation of the group at NBS-Washington. At that time we were at the beginning of modern numerical analysis, and so we were concerned with basic problems, such as, solutions of systems of linear equations, inversion of matrices, solutions of characteristic value (eigenvalue) problems, and related topics. There was much collaboration and even team work as the depth and magnitude of the problems became understood. There was an increasing amount of experimental computation as machines became available. There was considerable international cooperation, with surveys of the literature and translations when desirable. Work on many of these problems continues to this day. See, e.g., [11], [13].

This chapter is divided into three parts. In the first part we are concerned with properties of special matrices such as the Hilbert matrix. In the second part we discuss Gerschgorin theorems for characteristic values of matrices. In the final part we give a detailed account of the program at INA on solutions of systems of linear equations.

Hilbert Matrices and Related Topics

Historically, one of the first projects was the study of the Hilbert matrix. It all began with a letter, dated October 3, 1947, to Olga Todd from George Temple (in whose department at King's College, London, John Todd worked). He pointed out that the (British) Royal Aircraft Establishment (RAE) was interested in properties of the $n \times n$ Hilbert matrix

$$\mathbf{H}_n = [1/(i+j-1)] \qquad (i,j=1,2,\ldots,n)$$

which had been studied by Hilbert in 1894. W. W. Sawyer, R. A. Fairthorne and J. C. P. Miller had computed the dominant characteristic value λ_n of $\mathbf{H}_n$ for small values of n. In particular

$$\lambda_2 \approx 1.27, \quad \lambda_3 \approx 1.41, \quad \lambda_4 \approx 1.50, \quad \lambda_5 \approx 1.57$$

$$\lambda_6 \approx 1.62, \quad \lambda_8 \approx 1.70, \quad \lambda_{10} \approx 1.75, \quad \lambda_{20} \approx 1.91.$$

Olga Todd showed that the asymptotic behavior of λ_n is

$$\lambda_n = \pi + O(1/\log n).$$

She also considered the corresponding infinite Hilbert matrix $\mathbf{H} = \mathbf{H}_\infty$. It is known (see Hardy, Littlewood, Pólya: *Inequalities*) that the (infinite) quadratic form

$$Q(\mathbf{x}) = \Sigma\ \Sigma\ a_{ij} x_i x_j,$$

where $a_{ij} = 1/(i+j-1)$ and both summations are over the positive integers, satisfies the relation

$$Q(x) < \pi \Sigma\ x_i^2 \text{ when } 0 < \Sigma x_i^2 < \infty$$

and this result is the best possible. Thus there is no nonzero vector $\mathbf{x} = (x_1, x_2, \ldots)$ in l_2 (i.e., having

$\Sigma x_i^2 < \infty$) such that $\mathbf{Hx} = \pi\mathbf{x}$. She raised the question in the *Bulletin* of the AMS as to the existence of a vector $\mathbf{x}$, not in l_2, for which $\mathbf{Hx} = \pi\mathbf{x}$. We shall return to this point later.

The matrix $\mathbf{H}_n$ was notorious for its "ill condition," i.e., its inverse was hard to compute due to cancellations in the computation and to round-off. The matrix had turned up in practice, e.g., in least squares fitting of polynomials. The value of the determinant, det $\mathbf{H}_n$, was known to Cauchy and the exact inverse (all elements integral) had been computed by E. Lukacs and I. R. Savage [111, 105–108]. About this time a serious study of "ill-condition" was beginning, e.g., by Turing and by von Neumann and Goldstine. In this connection John Todd evaluated various condition-numbers of special matrices which occurred in practice. These condition-numbers were figures-of-merit which, when large, indicated that computational difficulties might be expected in inversion. He found that, while the condition-numbers of matrices associated with the discretization of differential equations were of polynomial growth in n, those for the Hilbert matrix were exponential. See AMS 29, 39, and a survey article "On Condition Numbers" by John Todd [118].

Other aspects of condition-numbers were studied at INA, such as, the effect of symmetrization on condition-numbers, by Olga Todd [110], an extension of Gauss's work on improving the condition of linear systems by G. E. Forsythe and T. S. Motzkin [132], and best conditioned matrices by G. E. Forsythe and E. G. Straus [23].

Because of this ill condition many variants of the Hilbert matrix have been used as test matrices for inversion programs and for programs for solving systems of linear equations.

We now return to the characteristic value/vector problem for the Hilbert matrix $\mathbf{H}_n$. The Perron-Frobenius theory of positive matrices guarantees the existence of a (strictly) dominant characteristic value λ_n, with a corresponding positive characteristic vector. These objects can therefore be calculated by the power method. G. E. Forsythe and M. Ascher computed the following values of λ_n on the SWAC

$$\lambda_{50} \approx 2.08, \quad \lambda_{65} \approx 2.11, \quad \lambda_{75} \approx 2.16$$

$$\lambda_{100} \approx 2.18, \quad \lambda_{125} \approx 2.21, \quad \lambda_{200} \approx 2.27$$

Other calculations were made by S. Schechter on a UNIVAC. This slow approach of λ_n to π was intriguing and, as Littlewood [57, vol. 1, 22] wrote (in the context of good approximation), "Few mathematicians can resist striking approximations even in a trivial context." There was therefore considerable activity, not only for the Hilbert matrix, but for the important class of Toeplitz matrices to which it belongs.

The Hilbert matrix is also a totally-positive matrix and as such its characteristic vectors exhibit a regular sign pattern: the dominant one has no change in sign, the next one has a single change in sign, the next one has two sign changes, and so on. This pattern can be used for testing programs for the solution of the complete characteristic value problem.

There was further work at RAE by a German aeronautical engineer, Peter F. Jordan. By interesting heuristic methods he obtained approximations to the characteristic values of $\mathbf{H}_n$. Despite various attempts no rigorous treatment was found. Jordan's conjectures are stated as Problem 23 in J. E. Littlewood's book *Some problems in real and complex analysis*, Heath 1965. Littlewood remarks: "All unrigorous but it is quite a problem." [It is stated in this reference that the conjectures were published in *Nature*, March 4, 1949, but we have not been able to trace this: perhaps Jordan's letter was submitted at that time.]

Contributions to the study of H were also made by W. Magnus at about this time. See items 25, 26 in [61].

Considerable time elapsed before further significant progress was made. The first advance was made at NBS and came to pass in the following way. There had been an interest in expanding the international visitor program from Europe to Asia. No suitable candidate was apparent until Olga Todd learned from George Temple of the work of Tosio Kato, a Japanese theoretical physicist, who had made novel estimates of the first sub-dominant characteristic value. After long negotiations Kato was appointed to NBS and attacked the problem. Kato has had a distinguished career in mathematics at UC-Berkeley after temporary appointments at the Courant Institute, NYU.

Kato's first result, depending on monotony considerations, was that there was indeed a nonzero vector $\mathbf{x}$ such that $\mathbf{Hx}=\pi\mathbf{x}$, thus answering the problem of Olga Todd. He later showed that, if $M(\theta)$ is the Hilbert bound of the infinite matrix

$$\mathbf{H}(\theta)=[1/(i+j-\theta)],$$

so that

$$M(\theta)=\pi \operatorname{cosec} \pi\theta,\ -2<\theta\leq-3/2,\quad M(\theta)=\pi,\ -3/2\leq\theta,$$

then every λ with $\lambda>M(\theta)$ is a characteristic value of $\mathbf{H}(\theta)$ with a corresponding positive characteristic vector and there is no characteristic value $\lambda<M(\theta)$ of $\mathbf{H}(\theta)$ with a positive characteristic vector.

The next development was due to Marvin Rosenblum, now at the University of Virginia, who first showed that every complex number with positive real part is a characteristic value of $\mathbf{H}(\theta)$, θ fixed and positive, by exhibiting the corresponding characteristic vector in terms of special functions. Rosenblum later determined the complete spectrum of $\mathbf{H}(\theta)$ using the Titchmarsh-Kodaira theory of singular differential operators.

The story ends with the work of N. G. DeBruijn and H. S. Wilf who showed, in 1966, that

$$\lambda_n=\pi-\tfrac{1}{2}\pi^5(\log n)^{-2}+O((\log\log n)(\log n)^{-3})$$

using results of Widom.

For a history of this problem to 1960 see John Todd, "Computational problems concerning the Hilbert matrix," [117]. For a systematic account of related matters see H. S. Wilf [127].

The Hilbert matrix continues to be an object of interest to various groups of mathematicians as evidenced by the recent paper by M. D. Choi, "Tricks or treats with the Hilbert matrix" [11].

Gerschgorin Theorems and Related Results

So far we have been concerned largely with the asymptotics of the dominant characteristic value of $\mathbf{H}_n$. We now turn our attention to less precise estimates of all the characteristic values of a matrix. It is important to have inclusion regions, that is, sets Σ of points in the Euclidean plane which include σ, the set of all the characteristic values of a matrix. In particular, it is often important to know when a matrix is convergent, i.e., when σ is in U, the unit disk $|z|<1$, or when it is stable, i.e., when σ is in L, the left-hand half plane $\mathrm{R}z<0$. These problems are, however, essentially equivalent since the Cayley map

$$w=(z-1)/(z+1)$$

transforms the interior of the unit circle into the left half plane.

We begin with the Gerschgorin theory. Recall that a matrix $\mathbf{M}=[m_{ij}]$ has a strictly dominant diagonal if

$$|m_{ii}|>\sum_j{}'|m_{ij}|,\quad \text{for all } i.$$

Here $\sum_j{}'$ excludes $j=i$. The fact that a matrix with a strictly dominant diagonal is nonsingular was known from the 1880's and is associated with many scientists—this is what Olga Todd called "the recurring theorem" in a *Monthly* article [108]. Applying this to the matrix $\mathbf{M}=\mathbf{A}-z\mathbf{I}$ shows that, if

$$|z-a_{ii}|>\sum_j{}'|a_{ij}|,\quad \text{all } i,$$

then $\mathbf{M}$ is nonsingular and z is not a characteristic value of $\mathbf{A}$. All the characteristic values of $\mathbf{A}$ therefore lie inside or on the boundary of the union of the n disks

$$|z-a_{ii}|\leq\sum_j{}'|a_{ij}|=r_i.$$

These are the Gerschgorin disks or circles; they form an inclusion region Σ. This theorem of Gerschgorin [25] had been pointed out to Olga Todd by N. Aronszajn. She had used this and various refinements of it in her war work (at NPL for the Ministry of Aircraft Production) in connection with flutter of aircraft. One of the refinements was to note that, under certain circumstances (irreducibility), it was not necessary to have strict diagonal dominance. Another was the remark that since a similarity does not change the characteristic values, new, in general different, inclusion regions can be obtained by applying the results to SAS^{-1} and using the fact that intersections of inclusion regions are again inclusion regions. Particularly important in practice is the case of diagonal similarities. An important fact is that if a connected subset of $r(< n)$ Gerschgorin disks is isolated from the complementary set, it necessarily contains r characteristic values. The use of diagonal similarities in the case $r = 1$ was exploited successfully by Olga Todd in practical cases and later studied theoretically by P. Henrici, John Todd [120], R. S. Varga [123], and others.

Among others connected with NBS or INA who worked in this general area were Ky Fan, A. J. Hoffman, and H. Wielandt. Hoffman, with Camion, proved a converse to the Gerschgorin theorem [34].

We discuss further representative results briefly. Since the characteristic values of a matrix and its transpose coincide, we can get column Gerschgorin circles with the same centers and (possibly different) column radii $c_j = \sum_i' |a_{ij}|$. Ostrowski mixed the rows and columns. He showed that the union of the circles with centers a_{ii} and radii $r_i^{\beta} c_i^{1-\beta}$ for any β, $0 \leq \beta \leq 1$ includes all the characteristic values of A. Instead of circles, A. T. Brauer [111, 101–106] used the Cassini Ovals defined by the relations $|z - a_{ii}|\,|z - a_{jj}| \leq r_i r_j$. Olga Todd prepared a bibliography on bounds for the characteristic values of finite matrices [105], which was much in demand.

We have already noted the valuable contributions made to the INA program by European and Asian visitors, thanks to the enlightened hiring policies of the NBS and the U.S. Civil Service Commission. There was a notable contribution from S. Africa. In the early post war years Philip Stein, together with a pupil, R. L. Rosenberg, submitted a paper to the London Mathematical Society. This paper was concerned with comparison of the two classical iterative methods for the solution of linear systems associated with the names of Jacobi and Gauss-Seidel. Olga Todd, as referee, noted its importance and novelty and gave them detailed advice and made sure that it was published. This paper [93] has become a classic and the Stein-Rosenberg Theory is a standard chapter in courses on iterative matrix analysis. During his visit to NBS and later, Stein worked on other problems suggested by Olga Todd, e.g., in Gerschgorin Theory and in Lyapunov Theory, with which his characterization of matrices C such that $C^r \to 0$, is closely connected.

R. S. Varga and his many students built up a large body of work in this area. There were many generalizations, e.g., to infinite dimensional problems, and developments from all over the world from the early 1950's to the present day.

Solutions of Systems of Linear Equations

One of the main projects pursued at INA was the study of methods for solving a system of linear equations by machine methods. A systematic study of this problem was undertaken in 1949 when Barkley Rosser became the Director of INA. It should be remembered that, in 1949, the only machine available to us was the IBM CPC. The SWAC was being built and could not be used. Of course, desk machines were available, but we were interested in using "automatic" computing machines. Apart from the CPC, we had to imagine what it would be like to use such machines. At that time (as well as at the present time) a standard method for solving a system of linear equations was some version of the Gaussian Elimination Method. Everett Yowell had already devised a satisfactory elimination procedure for the CPC. He pivoted by permutation of rows. Variants of his method are easily adapted to any "automatic" computing machine. For this reason we were concerned mainly with other methods for solving linear equations.

In 1949 Barkley Rosser organized a weekly seminar on the study of methods for solving systems of linear equations and for finding eigenvalues of matrices. The regular members attending this seminar were Barkley Rosser, Gertrude Blanch, George Forsythe, Magnus Hestenes, William

Karush, Cornelius Lanczos, and Marvin Stein. It was decided that Rosser and Forsythe should be responsible for the study of methods for solving linear equations. Hestenes, Karush, and Stein should emphasize methods for finding eigenvalues of matrices. Lanczos already had a program of study of methods for finding eigenvalues and eigenvectors of matrices and so would continue working on his program. Blanch was busy running the Computational Unit and acted as an advisor for the group. We also called upon Yowell for advice. All members of the group participated in both programs. Forsythe undertook the task of reviewing the literature on these subjects. This led to the publication (in AMS 29) of his important paper in which he classified methods for solving linear equations and gave an extensive bibliography of this subject.

Let us try to recall some of the ideas presented in this seminar. It will be convenient to present these ideas in a slightly more general form than they originally occurred to us. Consider the problem of finding the solution $\mathbf{x}$ of the linear equation

$$\mathbf{Ax}=\mathbf{k}, \tag{1}$$

where $\mathbf{A}$ is a nonsingular $n \times n$-matrix and $\mathbf{k}$ is an n-dimensional column vector. Its solution is

$$\mathbf{z}=\mathbf{A}^{-1}\mathbf{k}, \tag{2}$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. We assume that $\mathbf{A}^{-1}$ is unknown. For a given estimate $\mathbf{x}$ of $\mathbf{z}$, we use the size of the residual vector

$$\mathbf{r}=\mathbf{k}-\mathbf{Ax}=\mathbf{A}(\mathbf{z}-\mathbf{x}) \tag{3}$$

as a measure of the accuracy with which $\mathbf{x}$ estimates $\mathbf{z}$. Clearly, $\mathbf{x}=\mathbf{z}$ if and only if $\mathbf{r}=0$. We shall use the superscript T on vectors and matrices to denote the transpose. Thus, $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. Recall that $\mathbf{A}$ is said to be symmetric if $\mathbf{A}^T=\mathbf{A}$. We can measure the size of $\mathbf{r}$ by looking at its largest component or by looking at the length $|\mathbf{r}|$ of $\mathbf{r}$. Alternatively, we can look at the value of the function

$$F(\mathbf{x})=\tfrac{1}{2}\,\mathbf{r}^T\mathbf{Kr}=\tfrac{1}{2}\,\mathbf{x}^T\mathbf{Bx}-\mathbf{h}^T\mathbf{x}+\tfrac{1}{2}\mathbf{k}^T\mathbf{Kk}, \tag{4a}$$

where $\mathbf{K}$ is a positive definite symmetric matrix and

$$\mathbf{B}=\mathbf{A}^T\mathbf{KA}, \quad \mathbf{h}=\mathbf{A}^T\mathbf{Kk}. \tag{4b}$$

The matrix $\mathbf{B}$ is a positive definite symmetric matrix. When $\mathbf{K}=\mathbf{I}$, the identity, then $F=\frac{1}{2}|\mathbf{r}|^2$. The solution $\mathbf{x}=\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$ is the minimizer of F. The negative gradient of F is

$$\mathbf{g}=\mathbf{h}-\mathbf{Bx}=\mathbf{A}^T\mathbf{Kr}=\mathbf{B}(\mathbf{z}-\mathbf{x}). \tag{5}$$

Clearly $\mathbf{x}=\mathbf{z}$ if and only if $\mathbf{g}=0$, that is, if and only if $\mathbf{x}$ solves the equation $\mathbf{Bh}=\mathbf{h}$. The equation $\mathbf{Ax}=\mathbf{k}$ therefore can be replaced by an equation $\mathbf{Bx}=\mathbf{h}$ in which the matrix $\mathbf{B}$ is a positive definite symmetric matrix. Geometrically, the point $\mathbf{x}=\mathbf{z}$ is the center of the $(n-1)$-dimensional ellipsoid defined by the equation

$$F(\mathbf{x})=\gamma$$

where γ is a positive number. When $\mathbf{A}$ is a positive definite symmetric matrix, the choice $\mathbf{K}=\mathbf{A}^{-1}$ yields the relation $\mathbf{B}=\mathbf{A}$. In this case $\mathbf{x}$ solves $\mathbf{Ax}=\mathbf{k}$ if and only if $\mathbf{x}$ is the minimizer of the function

$$f(\mathbf{x})=\tfrac{1}{2}\mathbf{x}^T\mathbf{Ax}-\mathbf{k}^T\mathbf{x}+c,$$

where c is an arbitrary constant. The value of c is immaterial. For a given point $\mathbf{x}_1$, the $(n-1)$-dimensional ellipsoid E defined by the equation

$$f(\mathbf{x})=f(\mathbf{x}_1)$$

passes through $\mathbf{x}_1$ and the negative gradient $\mathbf{r}_1=\mathbf{k}-\mathbf{A}\mathbf{x}_1$ of f at $\mathbf{x}_1$ is an inner normal of E.

In our studies of iterative methods for solving $\mathbf{Ax}=\mathbf{k}$, we soon came to the conclusion that most of our iterations were equivalent to an iteration of the following type. Let $\mathbf{x}_1$ be an initial estimate of the solution $\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$. Having obtained an i-th estimate $\mathbf{x}_i$ of $\mathbf{z}$, compute an $(i+1)-$st estimate of $\mathbf{z}$ by a formula of the form

$$\mathbf{x}_{i+1}=\mathbf{x}_i-\mathbf{H}_i(\mathbf{A}\mathbf{x}_i-\mathbf{k})=\mathbf{x}_i+\mathbf{H}_i\mathbf{r}_i, \tag{6}$$

where the matrix $\mathbf{H}_i$ is determined by some rule. In many cases the matrix $\mathbf{H}_i$ is a fixed matrix $\mathbf{H}$. It is easily verified that the sequence $\{\mathbf{x}_i\}$ will converge linearly to $\mathbf{z}$ at a rate μ when

$$\mu=\limsup_{i\to\infty}\|\mathbf{I}-\mathbf{H}_i\mathbf{A}\|<1.$$

Here $\|\mathbf{M}\|$ is a norm of $\mathbf{M}$, which we choose to be the maximum of the length of $\mathbf{Mx}$ for all unit vectors $\mathbf{x}$.

There are a variety of routines for solving $\mathbf{Ax}=\mathbf{k}$ which are equivalent to an algorithm of the form (6) with $\mathbf{H}_i=\mathbf{H}$, a fixed matrix. One such routine proceeds as follows. We begin with an estimate $\mathbf{x}_1$ of the solution $\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$ together with a set of $m\geq n$ nonnull vectors $\mathbf{u}_1,\ldots,\mathbf{u}_m$ which span our space. We next select vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ such that the numbers

$$d_j=\mathbf{v}_j^{\mathrm{T}}\mathbf{A}\mathbf{u}_j \qquad (j=1,\ldots,m) \tag{7}$$

are not zero. For example, we can select $\mathbf{v}_j=\mathbf{KA}\mathbf{u}_j$, where $\mathbf{K}$ is a positive definite symmetric matrix. Having obtained an i-th estimate $\mathbf{x}_i$ of $\mathbf{z}$, we find an improved estimate $\mathbf{x}_{i+1}$ of $\mathbf{z}$ by the following subroutine.

Set $\mathbf{y}_1=\mathbf{x}_i$. Compute $\mathbf{y}_2,\ldots,\mathbf{y}_{m+1}$ successively by the formulas

$$\mathbf{y}_{j+1}=\mathbf{y}_j+a_j\mathbf{u}_j,\quad a_j=\mathbf{v}_j^{\mathrm{T}}(\mathbf{k}-\mathbf{A}\mathbf{y}_j)/d_j. \tag{8}$$

Then select $\mathbf{x}_{i+1}=\mathbf{y}_{m+1}$. Terminate the algorithm at the end of the i-th step if the residual $\mathbf{k}-\mathbf{A}\mathbf{x}_{i+1}$ is so small that $\mathbf{x}_{i+1}$ can be taken as a suitable estimate of the solution $\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$.

It should be noted that the scalar a_j in (8) is obtained by solving the equation

$$\mathbf{v}_j^{\mathrm{T}}[\mathbf{A}(\mathbf{y}_j+a_j\mathbf{u}_j)-\mathbf{k}]=\mathbf{v}_j^{\mathrm{T}}(\mathbf{A}\mathbf{y}_j-\mathbf{k})+a_jd_j=0, \tag{9}$$

that is, we choose a_j so that the residual $\mathbf{k}-\mathbf{A}\mathbf{y}_{j+1}$ is orthogonal to $\mathbf{v}_j$.

Observe that, because $\mathbf{k}=\mathbf{Az}$, Formula (8) can be rewritten in the form

$$\mathbf{y}_{j+1}-\mathbf{z}=\mathbf{W}_j(\mathbf{y}_j-\mathbf{z}), \tag{10}$$

where

$$\mathbf{W}_j=\mathbf{I}-\mathbf{u}_j\mathbf{v}_j^{\mathrm{T}}\mathbf{A}/d_j. \tag{11}$$

Setting

$$\mathbf{W}=\mathbf{W}_m\mathbf{W}_{m-1}\ldots\mathbf{W}_2\mathbf{W}_1 \tag{12}$$

we see, by (10) and the relations $\mathbf{x}_{i+1}=\mathbf{y}_{m+1}$ and $\mathbf{y}_1=\mathbf{x}_i$, that

$$\mathbf{x}_{i+1}-\mathbf{z}=\mathbf{W}_m\mathbf{W}_{m-1}\ldots\mathbf{W}_2\mathbf{W}_1(\mathbf{x}_i-\mathbf{z})=\mathbf{W}(\mathbf{x}_i-\mathbf{z}). \tag{13}$$

Defining

$$\mathbf{H}=(\mathbf{I}-\mathbf{W})\mathbf{A}^{-1} \tag{14}$$

we see that $\mathbf{W}=\mathbf{I}-\mathbf{HA}$ so that, with $\mathbf{Az}=\mathbf{k}$, (13) becomes

$$\mathbf{x}_{i+1}-\mathbf{z}=(\mathbf{I}-\mathbf{HA})(\mathbf{x}_i-\mathbf{z})=\mathbf{x}_i-\mathbf{z}-\mathbf{H}(\mathbf{Ax}_i-\mathbf{k})$$

so that our routine is equivalent to one of the form

$$\mathbf{x}_{i+1}=\mathbf{x}_i-\mathbf{H}(\mathbf{Ax}_i-\mathbf{k}), \tag{15}$$

as was proved.

Normally the rate of convergence is improved when we introduce a relaxation factor β so that Algorithm (8) takes the form

$$\mathbf{y}_{j+1}=\mathbf{y}_j+\beta a_j\mathbf{u}_j, \quad a_j=\mathbf{v}_j^{\mathrm{T}}(\mathbf{k}-\mathbf{Ay}_j)/d_j. \tag{16}$$

The formula for $\mathbf{W}_j$ is then

$$\mathbf{W}_j=\mathbf{I}-\beta\mathbf{u}_j\mathbf{v}_j^{\mathrm{T}}\mathbf{A}/d_j$$

and Algorithm (15) becomes

$$\mathbf{x}_{i+1}=\mathbf{x}_i-\mathbf{H}(\beta)(\mathbf{Ax}_i-\mathbf{k}), \tag{17}$$

where now $\mathbf{H}$ is a function of a positive number β. We have underrelaxation when $\beta<1$ and overrelaxation when $\beta>1$. In all cases we required β to be on the interval $0<\beta<2$. Incidentally it can be shown that, when we select $\mathbf{v}_j$ to be of the form $\mathbf{v}_j=\mathbf{KAu}_j$, Algorithm (17) will always converge when β is restricted in this manner.

In our seminar we did not begin with the general algorithm described above. We began with the (forward) Gauss-Seidel routine. This is the special case in which $m=n$ and $\mathbf{u}_j=\mathbf{v}_j=\mathbf{e}_j$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the unit coordinate vectors. In this event the matrix $\mathbf{H}(\beta)$ can be shown to be of the form

$$\mathbf{H}(\beta)=\beta(\mathbf{D}-\beta\mathbf{L})^{-1}, \tag{18}$$

where $\mathbf{D}$ is the diagonal matrix and $\mathbf{L}$ is the strictly lower triangular matrix which together with a strictly upper triangular matrix $\mathbf{U}$ are such that

$$\mathbf{A}=\mathbf{L}+\mathbf{D}+\mathbf{U}. \tag{19}$$

We also considered the (backward) Gauss-Seidel routine in which case $\mathbf{u}_j=\mathbf{v}_j=\mathbf{e}_{n-j+1}$ and

$$\mathbf{H}(\beta)=\beta(\mathbf{D}-\beta\mathbf{U})^{-1}. \tag{20}$$

Alternating the relaxed forward and backward Gauss-Seidel routines we obtain an Algorithm (17) having

$$\mathbf{H}(\beta)=\beta(2-\beta)(\mathbf{D}-\beta\mathbf{U})^{-1}\mathbf{D}(\mathbf{D}-\beta\mathbf{L})^{-1}. \tag{21}$$

This algorithm is equivalent to the routine generated by (8) when $m=2n-1$ and $\mathbf{u}_j=\mathbf{v}_j$ are chosen to be the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n, \mathbf{e}_{n-1}, \ldots, \mathbf{e}_2, \mathbf{e}_1$ successively. We normally chose $\beta=1$ in these three algorithms and did not attempt to find an optimal choice for β. Convergence is assured when $\mathbf{A}$ has sufficiently dominant main diagonal and when $\mathbf{A}$ is a positive definite symmetric matrix. In the study of relatively simple examples, we found that it was not unusual for the convergence number μ to be very close to one. Values of μ, such as $\mu=0.99$ and $\mu=0.9999$, appeared. This led us to conclude that we could not expect to have rapid convergence except in very favorable cases.

We also considered other special cases of Algorithm (8). In particular, we studied the case in which $m=n$, $\mathbf{v}_j=\mathbf{e}_j$, and $\mathbf{u}_j$ is the i-th column of $\mathbf{A}^T$. This is called the Kaczmarz routine and has a geometrical interpretation in terms of projections on the hyperplanes associated with the equation $\mathbf{Ax}=\mathbf{k}$. We omit this interpretation. By means of the transformation $\mathbf{x}=\mathbf{A}^T\mathbf{w}$, it is seen this routine is equivalent to the application of the Gauss-Seidel routine to the equation

$$\mathbf{AA}^T\mathbf{w}=\mathbf{k}.$$

Convergence is assured because the matrix $\mathbf{AA}^T$ is a positive definite symmetric matrix. Alternatively, if we select $\mathbf{u}_j=\mathbf{e}_j$ and $\mathbf{v}_j$ to be the j-ith column of $\mathbf{A}$ we obtain a convergent routine which is equivalent to applying the Gauss-Seidel algorithm to the equation

$$\mathbf{A}^T\mathbf{Ax}=\mathbf{A}^T\mathbf{k}.$$

In the general case with $m=n$, Algorithm (8) is equivalent to applying the Gauss-Seidel routine to the equation

$$\mathbf{V}^T\mathbf{AUw}=\mathbf{V}^T\mathbf{k}, \tag{22}$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the column vectors of $\mathbf{V}$, $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are the column vectors of $\mathbf{U}$, and $\mathbf{x}=\mathbf{Uw}$. When $\mathbf{U}$ and $\mathbf{V}$ are chosen so that the matrix

$$\mathbf{D}=\mathbf{V}^T\mathbf{AU} \tag{23}$$

is a diagonal matrix, the solution $\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$ is obtained in one cycle (in steps) of Algorithm (8). One can construct $\mathbf{u}$'s and $\mathbf{v}$'s of this type by a biorthogonalization process. However, such a process is basically equivalent to an elimination procedure. We did not pursue these ideas further at that time. However, we did study the extension of Algorithm (8) in which the $\mathbf{u}$'s formed an infinite sequence and discussed its properties including the question of convergence. In particular, at the suggestion of Lanczos, we considered the case in which $\mathbf{u}_1=\mathbf{k}$ and $\mathbf{u}_{i+1}=\mathbf{A}^i\mathbf{k}$ otherwise. In later years Tompkins made effective use of the Kaczmarz routine. We next turned our attention to the study of gradient methods for solving linear equations. Simultaneously, Rosser [83] devised a method for computing the exact inverse of a matrix with integer coefficients.

In the study of gradient methods, we first considered the case in which our matrix $\mathbf{A}$ was positive definite. Then the solution $\mathbf{x}=\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$ is the unique minimizer of the quadratic function

$$f(\mathbf{x})=\tfrac{1}{2}\,\mathbf{x}^T\mathbf{Ax}-\mathbf{k}^T\mathbf{x}. \tag{24}$$

The residual $\mathbf{r}=\mathbf{k}-\mathbf{Ax}$ is the negative gradient of f at $\mathbf{x}$, and is therefore in the direction of steepest descent. The *optimal gradient method* proceeds as follows:

Select an initial point $\mathbf{x}_1$ and compute $\mathbf{r}_1=\mathbf{k}-\mathbf{Ax}_1$. Having found $\mathbf{x}_i$ and $\mathbf{r}_i$ compute $\mathbf{x}_{i+1}$ and $\mathbf{r}_{i+1}$ by the formulas

$$\mathbf{x}_{i+1}=\mathbf{x}_i+a_i\mathbf{r}_i, \quad \mathbf{r}_{i+1}=\mathbf{r}_i-a_i\mathbf{Ar}_i, \tag{25a}$$

$$a_i=c_i/d_i, \quad c_i=|\mathbf{r}_i|^2, \quad d_i=\mathbf{r}_i{}^T\mathbf{Ar}_i. \tag{25b}$$

Terminate at the end of the i-th step if $\mathbf{r}_{i+1}$ is so small that $\mathbf{x}_{i+1}$ can be taken as a reasonable estimate of the solution $\mathbf{z}$.

This routine is called optimal because $t=a_i$ is the minimizer of f on the line $\mathbf{x}=\mathbf{x}_i+t\mathbf{r}_i$. However, we quickly learned that normally it was not the best gradient routine. Forsythe constructed a random 6×6 positive definite matrix A and a vector $\mathbf{k}=\mathbf{Az}$ with $\mathbf{z}$ prescribed. Starting with an arbitrary initial point $\mathbf{x}_1$ distinct from $\mathbf{z}$, he applied the optimal gradient routine to the system $\mathbf{Ax}=\mathbf{K}$. In each case he failed to obtain a good estimate of $\mathbf{z}$ after a reasonable number of steps. The iteration simply "bogged down." He then replaced equations (25a) by the relaxed equations

$$\mathbf{x}_{i+1}=\mathbf{x}_i+\beta a_i\mathbf{r}_i, \quad \mathbf{r}_{i+1}=\mathbf{r}_i-\beta a_i\mathbf{A}\mathbf{r}_i,$$

and obtained far better convergence when β was chosen to be one of the values 0.7, 0.8, or 0.9. The choice $\beta=1$, of course, gives us the original optimal gradient routine. Further experiments of this type were carried out by M. L. Stein with essentially the same results. Hestenes and Karush had similar experiences in the application of gradient methods to obtain eigenvalues of symmetric matrices. Motzkin suggested an alternative method for improving the optimal gradient routine. He suggested that an acceleration step be introduced at appropriate times. This step consisted of minimizing f on the line joining the points $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$. The algorithm was then restarted with this linear minimum point as the initial point. This too was successful. It was found that an acceleration routine of this type had been introduced earlier by A. C. Aitken. We tried minimizing f on 2-planes or 3-planes instead of on lines. But these routines were cumbersome and did not yield significantly better results. Hestenes and Stein prepared an extensive report of minimization methods for solving linear equations. This report remained unpublished until 1973 when it was published as a Historical Paper in the *Journal of Optimization Theory and Applications* [31]. Publication was held up because, in the meantime, Hestenes had devised an n-step conjugate gradient method which was based on the concepts of relative gradients and of conjugacy, concepts which he had used in his studies of variational theory. Hestenes and M. L. Stein established convergence properties for a large class of iterative processes in the real and complex cases, including a generalized gradient routine of the following type. As before, let A be a real positive definite matrix and let $f(\mathbf{x})$ be the associated quadratic function given by formula (24). Its minimum point $\mathbf{z}$ is the solution of $\mathbf{Ax}=\mathbf{k}$. Let $\mathbf{H}$ be a second positive definite symmetric matrix. We call the vector

$$\mathbf{g}=-\mathbf{Hr}=\mathbf{H}(\mathbf{Ax}-\mathbf{k})$$

a generalized gradient of f for reasons that will be explained below. The *generalized optimal gradient method* proceeds as follows: Choose an initial point $\mathbf{x}_1$ and perform the iteration defined by the equations

$$\mathbf{r}_i=\mathbf{k}-\mathbf{A}\mathbf{x}_i, \quad \mathbf{p}_i=\mathbf{H}\mathbf{r}_i \tag{26a}$$

$$a_i=c_i/d_i, \quad c_i=\mathbf{p}_i^{\mathrm{T}}\mathbf{r}_i, \quad d_i=\mathbf{p}_i^{\mathrm{T}}\mathbf{A}\mathbf{p}_i, \tag{26b}$$

$$\mathbf{x}_{i+1}=\mathbf{x}_i+a_i\mathbf{p}_i, \text{ that is, } \mathbf{x}_{i+1}=\mathbf{x}_i+a_i\mathbf{H}\mathbf{r}_i. \tag{26c}$$

Observe that, if $\mathbf{H}=\mathbf{A}^{-1}$, then $a_1=1$ and $\mathbf{x}_2=\mathbf{A}^{-1}\mathbf{k}$, giving us the solution in one step. This suggests that, if we have a rough estimate $\mathbf{H}$ of $\mathbf{A}^{-1}$, Algorithm (26) should be very effective. We did not test this conjecture numerically. To explain why we call $\mathbf{g}=-\mathbf{Hr}$ a generalized gradient of f, let us recall some concepts used by Hestenes in his studies of Variational Theory. Let $f(\mathbf{x})$ be a function on an inner product space possessing a directional derivative

$$f_1(\mathbf{x};\mathbf{y})=\left(\frac{d}{dt}\right)f(\mathbf{x}+t\mathbf{y})\big|_{t=0},$$

which is linear in $\mathbf{y}$. There is a vector $\mathbf{g}$, called a "generalized" gradient of f at $\mathbf{x}$, such that

$$f_1(\mathbf{x};\mathbf{y})=\langle\mathbf{g},\mathbf{y}\rangle$$

for all vectors y. Here $<z,y>$ is our inner product. In our case, with $<z,y> = z^T H^{-1} y$, we have

$$f_1(x;y) = (Ax - k)^T y = -r^T y = <g,y> = g^T H^{-1} y$$

so that $g = -Hr$ is our "generalized" gradient. When $H = I$, we have the usual gradient. M. L. Stein [91] used the generalized gradient concept in his Ph. D. dissertation to establish convergence theorems for gradients methods and Newton's method for variational problems. His thesis was sponsored by INA.

His study of the concepts of gradients and of conjugacy in variational theory led Hestenes to the development of the theory of conjugate gradients. Early in July 1951 he used these ideas to develop a conjugate gradient algorithm for solving $Ax = k$. In describing the method of conjugate gradients, we continue with the assumption that the matrix A in the equation $Ax = k$ is a positive definite symmetric matrix. Two vectors **p** and **q** are said to be conjugate if

$$p^T A q = 0.$$

The term "A-orthogonal" is also used for this concept. An i-dimensional plane π_i is conjugate to a vector **p** if **Ap** is normal to π_i or equivalently if every vector **q** in π_i is conjugate to **p**. On π_i the direction of steepest descent of

$$f(x) = \tfrac{1}{2} x^T A x - k^T x$$

at a point **x** in π_i is the negative gradient of f relative to π_i and is accordingly the orthogonal projection **p** on π_i of the negative gradient $r = k - Ax$ of f at **x**. For $\lambda > 0$, we call λp a "conjugate gradient" of f at **x** on π_i. The term "conjugate" is a geometrical term associated with ellipsoids. For example, in the two-dimensional case, the midpoints of parallel chords of an ellipse lie on a line "conjugate" to these chords and passing through the center for the ellipse, as shown in the following figure.

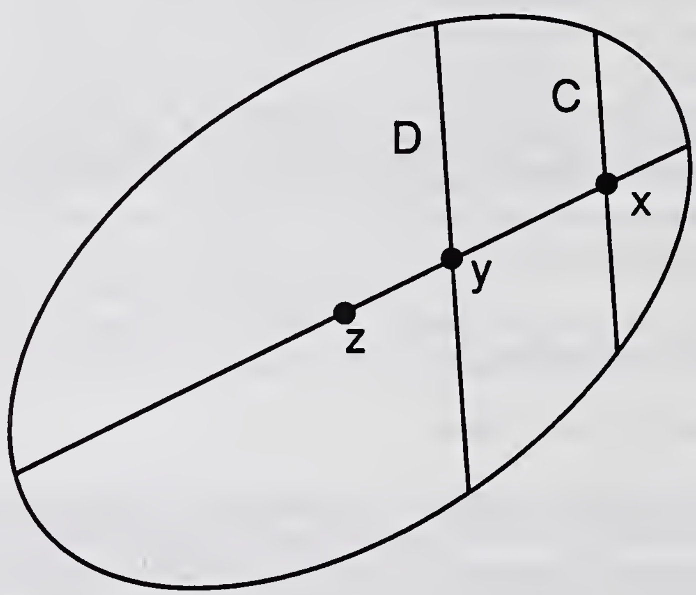

Figure 1

In this figure, **x** and **y** are respectively midpoints of parallel chords C and D. The chord through **x** and **y** is conjugate to C and D and has as its midpoint the center **z** of the ellipse. This result, which we have described for an ellipse, holds in general. The midpoints of parallel chords of an i-dimensional ellipsoid E_i lie in an i-plane π_i "conjugate" to these chords and passing through the center $\mathbf{z}$ of E_i. Moreover, if E_i is the intersection of the $(n-1)$-dimensional ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_i)$$

with an $(i+1)$-plane π_{i+1}, then $\mathbf{x}_i$ is on E_i and the orthogonal projection $\mathbf{p}_i$ on π_{i+1} of the negative gradient $\mathbf{r}_i=\mathbf{k}-\mathbf{A}\mathbf{x}_i$ of f at $\mathbf{x}_i$ is an inner normal of E_i at $\mathbf{x}_i$. This inner normal $\mathbf{p}_i$ gives the direction of steepest descent of f at $\mathbf{x}_i$ relative to the $(i+1)$-plane π_{i+1} and is the "conjugate gradient" of f at $\mathbf{x}_i$ on π_{i+1}.

To find the midpoint $\mathbf{m}$ of a chord C of E_i, observe that $\mathbf{m}$ is the minimum point of f on this chord and hence also on the line L containing the chord as a line segment. If $\mathbf{x}$ is a point on L and $\mathbf{p}$ is a direction vector for L, the point $\mathbf{x}+t\mathbf{p}$ is on L for all values of t. Moreover,

$$f(\mathbf{x}+t\mathbf{p})=f(\mathbf{x})-ct+\tfrac{1}{2}dt^2,$$

where

$$c=\mathbf{r}^{\mathrm{T}}\mathbf{p},\quad d=\mathbf{p}^{\mathrm{T}}\mathbf{A}\mathbf{p},\quad \mathbf{r}=\mathbf{k}-\mathbf{A}\mathbf{x}.$$

The minimizer of $f(\mathbf{x}+t\mathbf{p})$ with respect to t is $t=c/d$. Consequently, the point $\mathbf{m}=\mathbf{x}+a\mathbf{p}$, with $a=c/d$, is the minimum point of f on L and is therefore the midpoint of C.

The geometric algorithm for finding the center $\mathbf{z}$ of an ellipse E, shown in figure 1, can be put in algebraic form as follows. Referring to figure 2, let $\mathbf{x}_1$ be a point on an ellipse E and let $\mathbf{p}_1$ be a

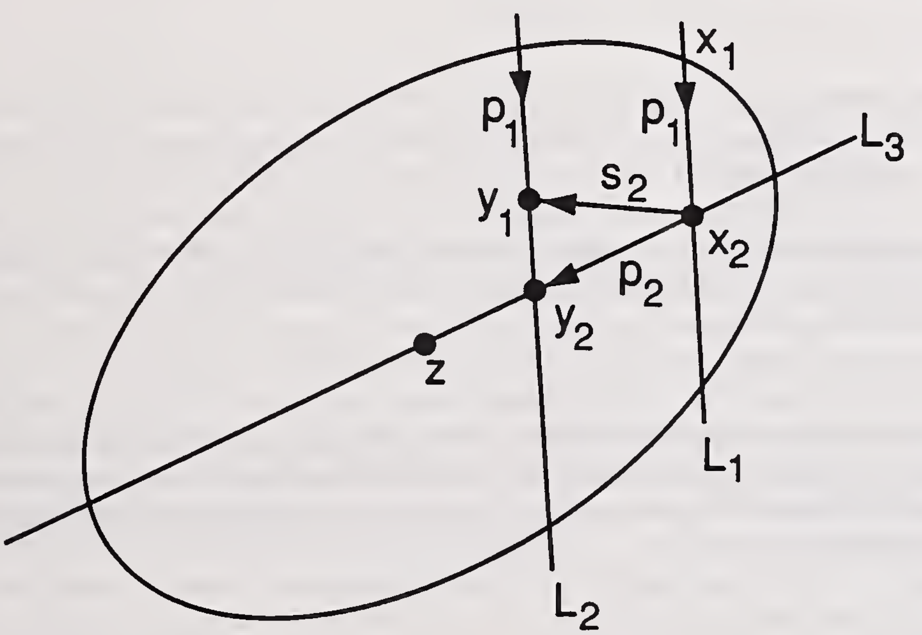

Figure 2

vector pointing towards the interior of E at $\mathbf{x}_1$. The minimizer $\mathbf{x}_2$ of f on the line

$$L_1:\quad \mathbf{x}=\mathbf{x}_1+t\mathbf{p}_1,$$

with t as a parameter, is given by the formula

$$\mathbf{x}_2=\mathbf{x}_1+a_1p_1,$$

where a_1 is chosen in the manner described above. Select a second point $\mathbf{y}_1=\mathbf{x}_2+\mathbf{s}_2$ which is not on L_1. The line

$$L_2: \quad \mathbf{y} = \mathbf{y}_1 + t\mathbf{p}_1$$

is parallel to L_1. The minimum point $\mathbf{y}_2$ of f on L_2 is

$$\mathbf{y}_2 = \mathbf{y}_1 + b_1\mathbf{p}_1$$

for a suitable choice of b_1. The vector

$$\mathbf{p}_2 = \mathbf{y}_2 - \mathbf{x}_2 = \mathbf{s}_2 + b_1\mathbf{p}_1$$

is conjugate to $\mathbf{p}_1$. The line

$$L_3: \quad \mathbf{x} = \mathbf{x}_2 + t\mathbf{p}_2$$

passes through $\mathbf{y}_2$. The minimum point

$$\mathbf{z} = \mathbf{x}_2 + a_2\mathbf{p}_2$$

of f on L_3 is the center $\mathbf{z}$ of the ellipse E. The method just described is a conjugate direction method for finding the center $\mathbf{z}$ of an ellipse E. When $\mathbf{p}_1$ is an inner normal of E at $\mathbf{x}_1$ and $\mathbf{s}_2$ is orthogonal to $\mathbf{p}_1$, this method becomes the conjugate gradient method for finding the center $\mathbf{z}$ of E. This choice of $\mathbf{p}_1$ and $\mathbf{s}_2$ can be made by selecting

$$\mathbf{p}_1 = \mathbf{r}_1 = \mathbf{k} - \mathbf{A}\mathbf{x}_1, \quad \mathbf{s}_2 = \mathbf{r}_2 = \mathbf{k} - \mathbf{A}\mathbf{x}_2.$$

In this event we have the formula

$$\mathbf{p}_2 = \mathbf{r}_2 + b_1\mathbf{p}_1, \quad b_1 = |\mathbf{r}_2|^2/|\mathbf{r}_1|^2$$

for the vector $\mathbf{p}_2$ conjugate to $\mathbf{p}_1$. The vector $\mathbf{p}_2$ is the A-orthogonal projection of $\mathbf{r}_2$ on the line L_3. It should be noted that, when $\mathbf{p}_1 = \mathbf{r}_1$, the line

$$L_4: \quad \mathbf{x} = \mathbf{x}_2 + t\mathbf{r}_2$$

is parallel to the tangent T of E at $\mathbf{x}_1$. It follows that the minimum point $\mathbf{z}_1$ of f on L_4 lies on the line L_5 joining $\mathbf{x}_1$ to $\mathbf{z}$. Thus, we can reach the center $\mathbf{z}$ of E by minimizing f successively on the lines L_1, L_4, and L_5, as pictured in figure 3. The points $\mathbf{x}_2$, $\mathbf{z}_1$, and $\mathbf{z}$ are the midpoints of the chords in which these lines cut the ellipse E.

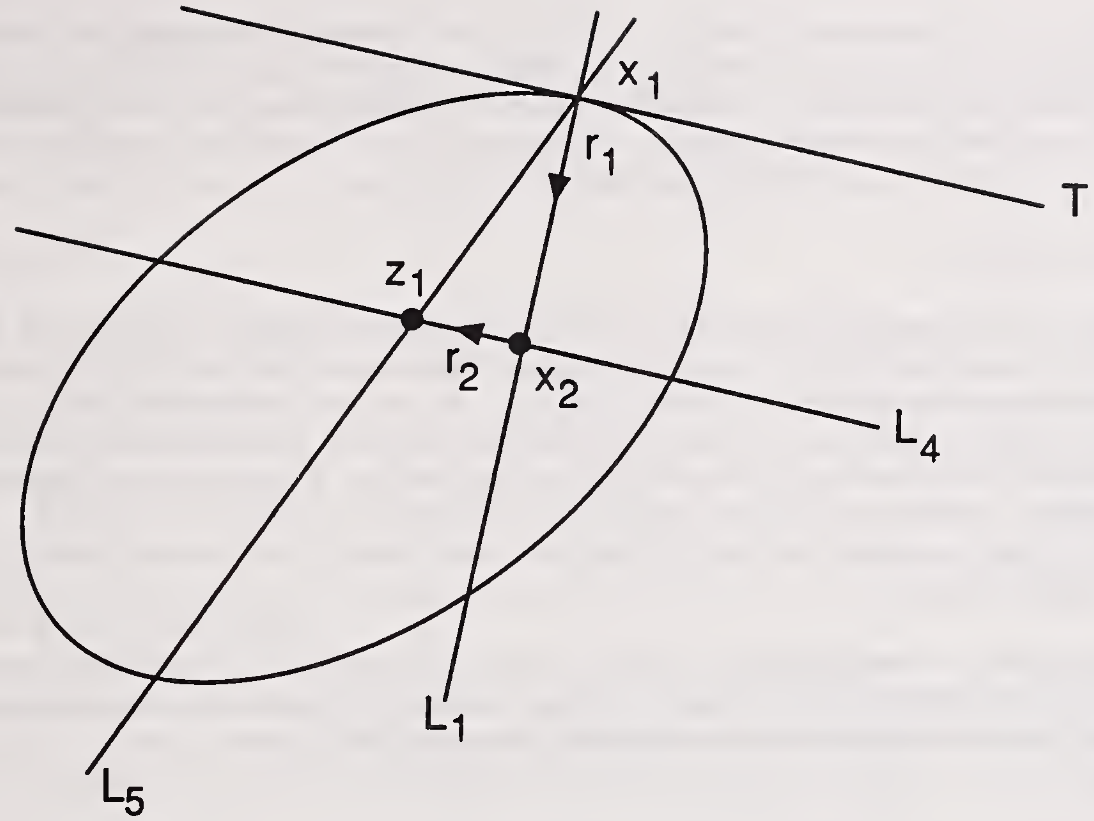

Figure 3

In the general case the method of conjugate gradients can be described as follows. In this description it will be convenient to omit all references to ellipsoids and their inner normals. This algorithm can be rephrased in terms of ellipsoids and will be done so later.

After selecting an initial point $\mathbf{x}_1$, we compute the steepest descent vector $\mathbf{p}_1$ of f at $\mathbf{x}_1$ and obtain the minimum point $\mathbf{x}_2$ of f on the line L_1 through $\mathbf{x}_1$ in the direction $\mathbf{p}_1$. The $(n-1)$-plane π_{n-1} through $\mathbf{x}_2$ conjugate to $\mathbf{p}_1$ contains the minimum point $\mathbf{z}$ of f, so that the dimension of our space of search has been reduced by one. We repeat the process restricting ourselves to the $(n-1)$-plane π_{n-1}. We select the steepest descent vector $\mathbf{p}_2$ of f at $\mathbf{x}_2$ in π_{n-1} and obtain the minimum point $\mathbf{x}_3$ of f on the line L_2 through $\mathbf{x}_2$ in the direction $\mathbf{p}_2$. The $(n-2)$-plane π_{n-2} in π_{n-1} through $\mathbf{x}_3$ and conjugate to $\mathbf{p}_2$ contains the minimum point $\mathbf{z}$ of f, so that at the next step we can limit our search to π_{n-2}, a space of one lower dimension. This process is continued, decreasing the dimension of our space of search by one in each step. In the i-th step we select a steepest descent vector $\mathbf{p}_i$ at $\mathbf{x}_i$ in an $(n-i+1)$-plane π_{n-i+1} and find the minimum point $\mathbf{x}_{i+1}$ of f on the line L_i through $\mathbf{x}_i$ in the direction $\mathbf{p}_i$. Our next space of search continuing $\mathbf{z}$ is the $(n-i)$-plane π_{n-i} in π_{n-i+1} through $\mathbf{x}_{i+1}$ and conjugate to $\mathbf{p}_i$. After $m \leq n$ steps we obtain a point $\mathbf{x}_{m+1}$ which coincides with the minimum point $\mathbf{z}$ of f, the solution $\mathbf{z}$ of $A\mathbf{x}=\mathbf{k}$.

The description of the conjugate gradient routine just given can be restated in terms of ellipsoids as follows: Select an initial point $\mathbf{x}_1$. Let E_{n-1} be the $(n-1)$-dimensional ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_1).$$

Its center $\mathbf{z}$ is the solution of $A\mathbf{x}=\mathbf{k}$. Let $\mathbf{p}_1$ be an inner normal of E_{n-1} at $\mathbf{x}_1$. Let $\mathbf{x}_2$ be the midpoint of the chord of E_{n-1} emanating from $\mathbf{x}_1$ in the direction of $\mathbf{p}_1$. The midpoints of the chords of E_{n-1} parallel to $\mathbf{p}_1$ determine an $(n-1)$-plane π_{n-1}, which contains the point $\mathbf{x}_2$ and the center $\mathbf{z}$ of E_{n-1}. The $(n-1)$-plane π_{n-1} intersects the ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_2)$$

in an $(n-2)$-dimensional ellipsoid E_{n-2} having $\mathbf{z}$ as its center. We repeat this process with E_{n-1} replaced by E_{n-2} and $\mathbf{x}_1$ replaced by $\mathbf{x}_2$. Select an inner normal $\mathbf{p}_2$ of E_{n-2} at $\mathbf{x}_2$. Let $\mathbf{x}_3$ be the midpoint of the chord of E_{n-2} emanating from $\mathbf{x}_2$ in the direction of $\mathbf{p}_2$. The midpoints of the chords of E_{n-2} parallel to $\mathbf{p}_2$ determine an $(n-2)$-plane π_{n-2} in π_{n-1} containing $\mathbf{x}_3$ and the center $\mathbf{z}$ of E_{n-2}. The $(n-2)$-plane π_{n-2} intersects the ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_3)$$

in an $(n-3)$-dimensional ellipsoid E_{n-3} passing through $\mathbf{x}_3$ and having $\mathbf{z}$ as its center. We next select an inner normal $\mathbf{p}_3$ of E_{n-3} at $\mathbf{x}_3$ and repeat the construction relative to E_{n-3}. Proceeding in this manner we arrive at the $(n-1)$-step at a point $\mathbf{x}_{n-1}$ on a one-dimensional ellipsoid E_1, an ellipse. We select an inner normal $\mathbf{p}_{n-1}$ of E_1 at $\mathbf{x}_{n-1}$ and find the midpoint $\mathbf{x}_n$ of the chord of E_1 emanating from $\mathbf{x}_{n-1}$ in the direction of $\mathbf{p}_{n-1}$. The midpoints of the chords of E_1 parallel to $\mathbf{p}_{n-1}$ determine a line π_1, a 1-plane, containing $\mathbf{x}_n$ and the center $\mathbf{z}$ of E_1. The line π_1 intersects the ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_n)$$

in two points, whose midpoint $\mathbf{x}_{n+1}$ is the center $\mathbf{z}$ of E_1 and of E_{n-1}, the solution $\mathbf{x}=\mathbf{z}$ of $\mathbf{Ax}=\mathbf{k}$.

The description of the conjugate gradient method given above is somewhat involved. Fortunately, in applications, we need not explicitly determine the planes $\pi_{n-1}, \pi_{n-2}, \ldots$ or the ellipsoids $E_{n-1}, E_{n-2}, \ldots$. All we need to know are the formulas

$$\mathbf{r}_{i+1}=\mathbf{k}-\mathbf{Ax}_{i+1}, \quad \mathbf{p}_{i+1}=\mathbf{r}_{i+1}+b_i\mathbf{r}_i, \quad b_i=|\mathbf{r}_{i+1}|^2/|\mathbf{r}_i|^2$$

determining the direction $\mathbf{p}_{i+1}$ of steepest descent of f at $\mathbf{x}_{i+1}$ on the $(n-i)$-plane π_{n-i} conjugate to the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_i$ previously chosen. It is interesting to note that the 2-plane π through $\mathbf{x}_{i+1}$, containing the vectors $\mathbf{p}_i$ and $\mathbf{r}_{i+1}$, cuts the $(n-1)$-dimensional ellipsoid

$$f(\mathbf{x})=f(\mathbf{x}_i)$$

in an ellipse having $\mathbf{p}_i$ as an inner normal at $\mathbf{x}_i$. Since π is conjugate to $\mathbf{p}_1, \ldots, \mathbf{p}_i$, the problem of finding $\mathbf{p}_{i+1}$ becomes a two-dimensional problem of the type described earlier.

In view of the formula for $\mathbf{p}_{i+1}$ given above, the conjugate gradient routine for solving $\mathbf{Ax}=\mathbf{k}$ can be stated as follows: Initially select a point $\mathbf{x}_1$. Compute $\mathbf{p}_1=\mathbf{r}_1=\mathbf{k}-\mathbf{Ax}_1$. Having obtained $\mathbf{x}_i$, $\mathbf{r}_i$, and $\mathbf{p}_i$ compute $\mathbf{x}_{i+1}$, $\mathbf{r}_{i+1}$, and $\mathbf{p}_{i+1}$ by the formulas

$$a_i=c_i/d_i, \quad d_i=\mathbf{p}_i^{\mathrm{T}}\mathbf{A}\mathbf{p}_i, \quad c_i=\mathbf{p}_i^{\mathrm{T}}\mathbf{r}_i \quad \text{or} \quad c_i=|\mathbf{r}_i|^2 \tag{27a}$$

$$\mathbf{x}_{i+1}=\mathbf{x}_i+a_i\mathbf{p}_i, \quad \mathbf{r}_{i+1}=\mathbf{r}_i-a_i\mathbf{A}\mathbf{p}_i \tag{27b}$$

$$b_i=-\mathbf{p}_i^{\mathrm{T}}\mathbf{A}\mathbf{r}_{i+1}/d_i \quad \text{or} \quad b_i=|\mathbf{r}_{i+1}|^2/|\mathbf{r}_i|^2 \tag{27c}$$

$$\mathbf{p}_{i+1}=\mathbf{r}_{i+1}+b_i\mathbf{p}_i \tag{27d}$$

Terminate at the m-th step if $\mathbf{r}_{m+1}=0$. Then $m\leq n$ and $\mathbf{x}_{m+1}=\mathbf{z}$, the solution of $\mathbf{Ax}=\mathbf{k}$ and the minimum point of f.

Observe that we have given alternative formulas for b_i and c_i. It can be shown that the residuals $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are mutually orthogonal and that the conjugate gradients $\mathbf{p}_1, \ldots, \mathbf{p}_m$ are mutually conjugate. Also the residual $\mathbf{r}_{i+1}$ is orthogonal to the vectors $\mathbf{p}_1, \ldots, \mathbf{p}_i$. In addition

$$\mathbf{p}_i/c_i=\mathbf{r}_1/c_1+\mathbf{r}_2/c_2+\ \ldots\ +\mathbf{r}_i/c_i.$$

The vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ satisfy the relations

$$\mathbf{p}_2=(1+b_1)\mathbf{p}_1-a_1\mathbf{A}\mathbf{p}_1, \tag{28a}$$

$$\mathbf{p}_{i+1}=(1+b_i)\mathbf{p}_i-a_i\mathbf{A}\mathbf{p}_i-b_{i-1}\mathbf{p}_{i-1} \quad (i>1). \tag{28b}$$

Similarly, the residuals $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ satisfy the ternary relations

$$\mathbf{r}_2=\mathbf{r}_1-a_1\mathbf{A}\mathbf{r}_1, \tag{29a}$$

$$\mathbf{r}_{i+1}=(1+\bar{b}_{i-1})\mathbf{r}_i-a_i\mathbf{A}\mathbf{r}_i-\bar{b}_{i-1}\mathbf{r}_{i-1} \quad (i>1), \tag{29b}$$

where

$$\bar{b}_{i-1}=a_ib_{i-1}/a_{i-1}. \tag{29c}$$

In his original paper Hestenes used these ternary relations to give an alternative version of the conjugate gradient algorithm. It is now known as Gradient Partan. It is obtained by rewriting equations (29) in the following form

$$\mathbf{r}_2=\mathbf{r}_1-\beta_1\mathbf{A}\mathbf{r}_1, \tag{30a}$$

$$\mathbf{r}_{i+1}=(\mathbf{r}_i-\beta_i\mathbf{A}\mathbf{r}_i-\delta_{i-1}\mathbf{r}_{i-1})/(1-\delta_{i-1}), \tag{30b}$$

where

$$\beta_i=|\mathbf{r}_i|^2/\mathbf{r}_i^{\mathrm{T}}\mathbf{A}\mathbf{r}_i, \quad \delta_{i-1}=\mathbf{r}_{i-1}^{\mathrm{T}}(\mathbf{r}_i-\beta_i\mathbf{A}\mathbf{r}_i)/|\mathbf{r}_{i-1}|^2. \tag{30c}$$

The corresponding formulas for the points $\mathbf{x}_i$ are

$$\mathbf{x}_2=\mathbf{x}_1+\beta_1\mathbf{r}_1, \tag{30d}$$

$$\mathbf{x}_{i+1}=(\mathbf{x}_i+\beta_i\mathbf{r}_i-\delta_{i-1}\mathbf{x}_{i-1})/(1-\delta_{i-1}). \tag{30e}$$

The coefficients β_i and δ_{i-1} are chosen so that $\mathbf{r}_{i+1}$ is orthogonal to $\mathbf{r}_i$ and $\mathbf{r}_{i-1}$ respectively. As a result the residuals $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_i$ are mutually orthogonal so that i cannot exceed n. Algorithm (30) can be rewritten in the form

Select an initial point $\mathbf{x}_1$ and compute

$$\mathbf{r}_1=\mathbf{k}-\mathbf{A}\mathbf{x}_1, \quad \beta_1=|\mathbf{r}_1|^2/\mathbf{r}_1^{\mathrm{T}}\mathbf{A}\mathbf{r}_1, \tag{31a}$$

$$\mathbf{x}_2=\mathbf{x}_1+\beta_1\mathbf{r}_1, \quad \mathbf{r}_2=\mathbf{r}_1-\beta_1\mathbf{A}\mathbf{r}_1. \tag{31b}$$

For $i=2, 3, \ldots$ compute

$$\beta_i=|\mathbf{r}_i|^2/\mathbf{r}_i^{\mathrm{T}}\mathbf{A}\mathbf{r}_i, \tag{31c}$$

$$\mathbf{y}_{i+1}=\mathbf{x}_i+\beta_i\mathbf{r}_i, \quad \mathbf{s}_{i+1}=\mathbf{r}_i-\beta_i\mathbf{A}\mathbf{r}_i, \tag{31d}$$

$$\delta_{i-1}=\mathbf{r}_{i-1}^{\mathrm{T}}\mathbf{s}_{i+1}/|\mathbf{r}_{i-1}|^2, \tag{31e}$$

$$\mathbf{x}_{i+1}=(\mathbf{y}_{i+1}-\delta_{i-1}\mathbf{x}_{i-1})/(1-\delta_{i-1}), \tag{31f}$$

$$\mathbf{r}_{i+1}=(\mathbf{s}_{i+1}-\delta_{i-1}\mathbf{r}_{i-1})/(1-\delta_{i-1}). \tag{31g}$$

The point $\mathbf{y}_{i+1}$ is obtained from $\mathbf{x}_i$ by a gradient minimization step. The point $\mathbf{x}_{i+1}$ is obtained by minimizing f on the line joining $\mathbf{x}_{i-1}$, and $\mathbf{y}_{i+1}$. The last operation is an acceleration procedure of the type suggested by Motzkin. In this form the conjugate gradient routine consists of alternating a gradient minimization step and an acceleration step.

There is another set of points $\mathbf{z}_1=\mathbf{x}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$ associated with the conjugate algorithm (27). Observe that, for $i \leq m$, the vector $\mathbf{p}_i$ is of the form

$$\mathbf{p}_1=\mathbf{k}-\mathbf{A}\mathbf{z}_1, \tag{32a}$$

$$\mathbf{p}_i=m_i(\mathbf{k}-\mathbf{A}\mathbf{z}_i), \tag{32b}$$

$$\mathbf{p}_{i+1}=\mathbf{r}_{i+1}+b_i\mathbf{p}_i=m_{i+1}(\mathbf{k}-\mathbf{A}\mathbf{z}_{i+1}) \tag{32c}$$

so that

$$m_{i+1}=1+b_i m_i, \quad m_1=1, \tag{32d}$$

$$\mathbf{z}_1=\mathbf{x}_1, \quad \mathbf{z}_{i+1}=(\mathbf{x}_{i+1}+b_i m_i \mathbf{z}_i)/m_{i+1}. \tag{32e}$$

The conjugate gradient $\mathbf{p}_i$ is therefore a scaled negative gradient of f at the point $\mathbf{z}_i$. In addition, it can be shown that the point $\mathbf{z}_{i+1}$ minimizes the square $|\mathbf{r}|^2$ of the negative gradient $\mathbf{r}$ of f on the i-plane passing through the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i+1}$.

So far we have assumed that the matrix $\mathbf{A}$ was a positive definite matrix. The conjugate gradient routine is valid when $\mathbf{A}$ is a nonnegative symmetric matrix. In this case the algorithm terminates when either $\mathbf{r}_m=0$ or when $\mathbf{A}\mathbf{p}_m=0$. In the first case $\mathbf{x}_m$ solves $\mathbf{A}\mathbf{x}=\mathbf{k}$. In the second case $\mathbf{z}_m$ is a least square solution of $\mathbf{A}\mathbf{x}=\mathbf{k}$. It is the shortest least square solution when $\mathbf{x}_1=0$ is the initial point. The conjugate gradient routine also solves $\mathbf{A}\mathbf{x}=\mathbf{k}$ when $\mathbf{A}$ is symmetric and indefinite unless one encounters the situation in which $d_i=0$ before the solution is obtained. To insure the effectiveness of our algorithm, we excluded the case in which $\mathbf{A}$ is indefinite.

To solve the equation $\mathbf{A}\mathbf{x}=\mathbf{k}$ for an arbitrary matrix $\mathbf{A}$, we used a least square routine. That is, we applied the conjugate gradient algorithm to the equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{A}\mathbf{x}=\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{k}, \tag{33}$$

where $\mathbf{K}$ is a positive definite symmetric matrix. Initially, we chose $\mathbf{K}=\mathbf{I}$, the identity. The associated function to be minimized is

$$F(\mathbf{x})=\mathbf{r}^{\mathrm{T}}\mathbf{K}\mathbf{r}=(\mathbf{k}-\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{K}(\mathbf{k}-\mathbf{A}\mathbf{x}). \tag{34}$$

$$\mathbf{g}=\mathbf{A}^{\mathrm{T}}\mathbf{K}(\mathbf{k}-\mathbf{A}\mathbf{x}). \tag{35}$$

Anticipating a generalization made later by Hestenes, we introduce a generalized negative gradient $\mathbf{H}\mathbf{g}$ of F, where $\mathbf{H}$ is a positive definite symmetric matrix. Using generalized gradients, the conjugate gradient routine (27) for F can be put in the form:

Select an initial point $\mathbf{x}_1$ and perform the iteration defined by the following formulas

$$\mathbf{r}_1=\mathbf{k}-\mathbf{A}\mathbf{x}_1, \quad \mathbf{g}_1=\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{r}_1, \quad \mathbf{p}_1=\mathbf{H}\mathbf{g}_1, \tag{36a}$$

$$c_i=\mathbf{p}_i{}^{\mathrm{T}}\mathbf{g}_i, \quad \mathbf{s}_i=\mathbf{A}\mathbf{p}_i, \quad d_i=\mathbf{s}_i{}^{\mathrm{T}}\mathbf{K}\mathbf{s}_i, \quad a_i=c_i/d_i, \tag{36b}$$

$$\mathbf{x}_{i+1}=\mathbf{x}_i+a_i\mathbf{p}_i, \quad \mathbf{r}_{i+1}=\mathbf{r}_i-a_i\mathbf{s}_i, \quad \mathbf{g}_{i+1}=\mathbf{A}^{\mathrm{T}}\mathbf{K}\mathbf{r}_{i+1}, \tag{36c}$$

$$\mathbf{p}_{i+1}=\mathbf{H}\mathbf{g}_{i+1}+b_i\mathbf{p}_i, \quad b_i=\mathbf{g}_{i+1}{}^{\mathrm{T}}\mathbf{H}\mathbf{g}_{i+1}/c_i. \tag{36d}$$

The generalized gradients $g_1, g_2, \ldots$ are H-orthogonal and the conjugate gradients $p_1, p_2, \ldots$ are B-orthogonal, where $B=A^TKA$. It can be shown that, if $x_1, x_2, \ldots, x_{n+1}=z$ are the vertices of a nondegenerate simplex, matrices H and K can be chosen so that Algorithm (36) with x_1 as its initial point reproduces the points $x_1, x_2, \ldots, x_{n+1}=z$ in the given order. It follows that the estimates of z generated by an n-step algorithm can be reproduced by a conjugate gradient algorithm of the type (36), provided that these estimates are independent.

When A is a positive definite symmetric matrix, the choice $K=A^{-1}$, $H=I$ in (36) yields the original conjugate gradient Algorithm (27). The various versions of the conjugate gradient algorithm (with $K=H=I$) given above can be found in the original 1951 NBS report by Hestenes, in which he also discussed the complex case. This report remained unpublished until 1973 at which time it was published as a Historical Paper [29] in the *Journal of Optimization Theory and Applications*. It happened that Eduard Stiefel of Zurich, Switzerland had devised the conjugate gradient algorithm independently at about the same time or perhaps even a little earlier. When he arrived at UCLA to present his results at a Conference on Solutions of Linear Equations, he was given the NBS report by Hestenes and found that we at INA had devised the same algorithm. Because of this situation, it was decided that we should present our results in a joint paper. This led to the extensive report by Stiefel and Hestenes on Conjugate Gradients published in 1952 [30].

It should be noted that the ternary relations (29) for the residuals are equivalent to the ternary relations devised by Lanczos in his program for finding the eigenvalues of matrices. Lanczos presented these relations in the seminar sponsored by Rosser. It occurred to none of us at that time that these relations could be used effectively in an algorithm for solving linear equations in n steps. We were not aware of this connection until the conjugate gradient routine had been devised by geometrical considerations. It is clear therefore that the conjugate gradient algorithm is an easy consequence of results given by Lanczos. This led Lanczos to devise an alternative version of the conjugate gradient algorithm, which he called a "Method of Minimized Iterations." This method was published [51] in 1952 shortly before the joint paper of Stiefel and Hestenes appeared in the same journal. Consider the case in which the matrix A is a positive definite symmetric matrix. Then the routine proposed by Lanczos for solving $Ax=k$ proceeds as follows:

Select $s_1=q_1=k$. Then compute vectors $s_2, s_3, \ldots$ and $q_2, q_3, \ldots$ successively by the following algorithm

$$s_{i+1}=\beta_i s_i+Aq_i, \quad q_{i+1}=\sigma_i q_i+s_{i+1}, \tag{37a}$$

$$\gamma_i=|s_i|^2, \; \delta_i=s_i^TAq_i, \quad \beta_i=-\delta_i/\gamma_i \tag{37b}$$

$$\sigma_i=-\gamma_{i+1}/\delta_i, \quad \mu_0=1, \quad \mu_i=\mu_{i-1}/\beta_i. \tag{37c}$$

Terminate when $s_{m+1}=0$. Then $m\leq n$ and the solution $x=z$ of $Ax=k$ is

$$z=-\mu_1q_1-\mu_2q_2-\ \ldots\ -\mu_mq_m. \tag{37d}$$

Algorithm (37) is connected with the conjugate gradient Algorithm (27) by the relations

$$r_i=\mu_{i-1}s_i, \quad p_i=\mu_{i-1}q_i, \quad x_1=0, \tag{38a}$$

$$a_i=-1/\beta_i=-\mu_i/\mu_{i-1}, \quad b_i=\sigma_i/\beta_i \tag{38b}$$

Algorithm (37) is called a method of minimized iterations because s_{i+1} is the shortest vector expressible in the form

$$s_{i+1}=(m_0+m_1A+\ \ldots\ +m_{i-1}A^{i-1}+A^i)k. \tag{40}$$

One obtains the ternary relations of Lanczos by eliminating the q's in the relations (37). In his paper

Lanczos gave a detailed account of procedures for reducing the effects of round-off errors, including a discussion of how to employ Chebyshev polynomials for this purpose. His papers on this and related subjects are very significant and contain many practical suggestions. Lanczos is widely quoted by researchers in this field.

As suggested by Formula (40) for $\mathbf{s}_{i+1}$, every version of the conjugate gradient routine has associated with it a significant set of polynomials. For example, in the conjugate routine (27), the residuals $\mathbf{r}_i$ and conjugate gradients $\mathbf{p}_i$ are expressible in the form

$$\mathbf{r}_i = R_{i-1}(\mathbf{A})\mathbf{k}, \ \mathbf{p}_i = P_{i-1}(\mathbf{A})\mathbf{k}, \tag{41}$$

where $R_{i-1}(\lambda)$ and $P_{i-1}(\lambda)$ are polynomials in λ of degree $i-1$ having $R_{i-1}(0) = P_{i-1}(0) = 1$. These polynomials are generated by the algorithm

$$R_0 = P_0 = 1 \tag{42a}$$

$$R_i = R_{i-1} - a_i \lambda P_{i-1}, \quad P_i = R_i + b_i P_{i-1}, \tag{42b}$$

where $a_1, \ldots, a_m$ and $b_1, \ldots, b_{m-1}$ are the numbers appearing in Algorithm (27). The polynomial $R_m(\lambda)$ is a factor of the characteristic polynomial of A and is the characteristic polynomial of A when $m = n$. These polynomials therefore can be used to find the eigenvalues of A in the manner developed by Lanczos [50] or by some other means.

The conjugate gradient routine belongs to a class of routines which we call **conjugate direction routines.** In these routines we have given or we generate a set of n mutually conjugate vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ by some means. Then the solution $\mathbf{x} = \mathbf{z}$ of $\mathbf{Ax} = \mathbf{k}$ can be obtained by the algorithm

$$\mathbf{x}_1 \text{ arbitrary}, \quad \mathbf{r}_1 = \mathbf{k} - \mathbf{A}\mathbf{x}_1, \tag{43a}$$

$$a_i = c_i / d_i, \quad d_i = \mathbf{p}_i^T \mathbf{A} \mathbf{p}_i, \quad c_i = \mathbf{p}_i^T \mathbf{r}_i = \mathbf{p}_i^T \mathbf{r}_1, \tag{43b}$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i + a_i \mathbf{p}_i, \quad \mathbf{r}_{i+1} = \mathbf{r}_i - a_i \mathbf{A} \mathbf{p}_i. \tag{43c}$$

The vector $\mathbf{x}_{n+1}$ is the solution $\mathbf{z}$ of $\mathbf{Ax} = \mathbf{k}$. In addition the inverse of A is given by the formula

$$\mathbf{A}^{-1} = \mathbf{p}_1 \mathbf{p}_1^T / d_1 + \mathbf{p}_2 \mathbf{p}_2^T / d_2 + \ \ldots \ + \mathbf{p}_n \mathbf{p}_n^T / d_n. \tag{44}$$

It is generated by the iteration

$$\mathbf{B}_0 = 0, \quad \mathbf{B}_i = \mathbf{B}_{i-1} + \mathbf{p}_i \mathbf{p}_i^T / d_i, \quad i = 1, \ldots, n \tag{45}$$

We have $\mathbf{B}_n = \mathbf{A}^{-1}$. Mutually conjugate vectors are generated by the conjugate gradient algorithm (27). If we construct mutually conjugate vectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ in the following manner, we obtain a new routine, which we call a **Conjugate Gram Schmidt Process.**

Select a set of linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Then compute $\mathbf{p}_1, \ldots, \mathbf{p}_n$ as follows

$$\mathbf{p}_1 = \mathbf{u}_1, \tag{46a}$$

$$\mathbf{p}_{i+1} = \mathbf{u}_{i+1} - b_{i1} \mathbf{p}_1 - b_{i2} \mathbf{p}_2 - \ \ldots \ - b_{ii} \mathbf{p}_i, \tag{46b}$$

for $i = 1, \ldots, n-1$, where, for $j = 1, \ldots, i$,

$$b_{ij} = \mathbf{p}_j^T \mathbf{A} \mathbf{u}_{i+1} / d_j, \quad d_j = \mathbf{p}_j^T \mathbf{A} \mathbf{p}_j = \mathbf{p}_j^T \mathbf{A} \mathbf{u}_j. \tag{46c}$$

Introducing the commuting matrices

$$\mathbf{W}_i = \mathbf{I} - \mathbf{p}_i\mathbf{p}_i^T\mathbf{A}/d_i \quad \text{for } i = 1, \ldots, n, \tag{47a}$$

$$\mathbf{C}_0 = \mathbf{I},\ \mathbf{C}_i = \mathbf{W}_i\mathbf{C}_{i-1} \quad \text{for } i = 1, \ldots, n, \tag{47b}$$

we see that Algorithm (46) can be put in the matrix form

$$\mathbf{p}_i = \mathbf{C}_{i-1}\mathbf{u}_i \tag{48}$$

for $i = 1, \ldots, n$. Algorithm (46) can be rewritten in many ways and was done so by Stiefel and Hestenes [30]. They showed that, when we choose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ to be the unit coordinate vectors, Algorithm (46)–(43) is equivalent to a Gaussian Elimination method. Consequently, when $\mathbf{A}$ is a positive definite symmetric matrix, a version of the Gauss Elimination for solving $\mathbf{Ax} = \mathbf{h}$ can be viewed to be a Conjugate Gram Schmidt Process. It should be noted that, in Algorithm (46)–(43), the vector $\mathbf{u}_i$ need not be chosen before the i-th step. If, in the i-th step, we choose $\mathbf{u}_i$ to be the residual $\mathbf{r}_i$, our algorithm becomes a conjugate gradient algorithm with a "built in" procedure for reducing round-off errors and reduces to the conjugate gradient algorithm (27) when no round-off errors occur.

The Conjugate Gram Schmidt Algorithm and hence also Gaussian Elimination can be approached from a geometrical point of view. We term this geometrical approach, the **method of parallel displacements,** for reasons that will become evident. We proceed with the minimization with a positive definite quadratic function

$$f(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{k}^T\mathbf{x} + \mathbf{c}$$

as follows. We begin with $n+1$ independent points $\mathbf{x}_1, \mathbf{x}_{11}, \mathbf{x}_{21}, \ldots, \mathbf{x}_{n1}$, that is, $n+1$ points which do not lie in an $(n-1)$-plane. Equivalently, we begin with a point $\mathbf{x}_1$ and n linearly independent vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and set $\mathbf{x}_{i1} = \mathbf{x}_1 + \mathbf{u}_i$. Having obtained these points, we set $\mathbf{p}_1 = \mathbf{x}_{11} - \mathbf{x}_1$ and minimize f on the lines $\mathbf{x} = \mathbf{x}_{j1} + t\mathbf{p}_1$ for $j = 1, \ldots, n$. This yields n points $\mathbf{x}_2 = \mathbf{x}_{12}, \mathbf{x}_{22}, \ldots, \mathbf{x}_{n2}$, which lie in an $(n-1)$-plane π_{n-1} conjugate to $\mathbf{p}_1$ and containing the minimum point $\mathbf{z}$ of f. We have thereby reduced our space of search by one. Accordingly, we repeat the process in the $(n-1)$-plane π_{n-1}. We set $\mathbf{p}_2 = \mathbf{x}_{22} - \mathbf{x}_2$ and minimize f on the parallel lines $\mathbf{x} = \mathbf{x}_{j2} + t\mathbf{p}_2$ for $j = 2, \ldots, n$. This yields $n-1$ points $\mathbf{x}_3 = \mathbf{x}_{23}, \mathbf{x}_{33}, \ldots, \mathbf{x}_{n3}$ which lie in an $(n-2)$-plane π_{n-2} in π_{n-1} conjugate to $\mathbf{p}_2$ and containing the minimum point $\mathbf{z}$ of f. Because π_{n-2} is in π_{n-1}, it is conjugate to $\mathbf{p}_1$ as well as to $\mathbf{p}_2$. Proceeding in this way we find, in the i-th step, that we have $n-i+2$ points $\mathbf{x}_i, \mathbf{x}_{ii}, \ldots, \mathbf{x}_{ni}$ points lying in an $(n-i+1)$-plane π_{n-i+1} which contains the minimum point $\mathbf{z}$ of f and is conjugate to the vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}$. We continue by setting $\mathbf{p}_i = \mathbf{x}_{ii} - \mathbf{x}_i$ and minimizing f on the parallel lines $\mathbf{x} = \mathbf{x}_{ji} + t\mathbf{p}_i$ for $j = i, \ldots, n$. This yields $n-i+1$ points determining an $(n-1)$-plane π_{n-i} containing $\mathbf{z}$. The point $\mathbf{x}_{n+1}$ is the minimum point $\mathbf{z}$ of f. The vectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ are mutually conjugate.

The case $n = 3$ is shown schematically in the following diagram.

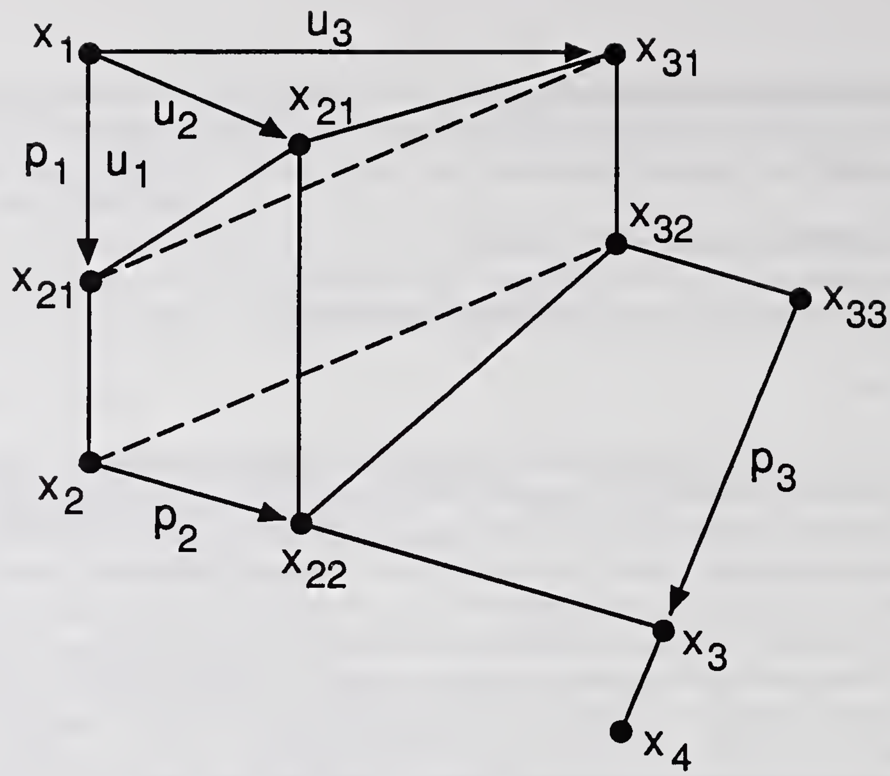

Figure 4

The geometrical procedure just described follows readily from the formulas given by Stiefel and Hestenes, although they did not explicitly give this interpretation.

When the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are mutually orthogonal, the Conjugate Gram Schmidt Algorithm (46) can be put in the following matrix form. Select $\mathbf{G}_0 = \mathbf{I}$, the identity, and perform the following iteration

$$\mathbf{p}_i = \mathbf{G}_{i-1}\mathbf{u}_i, \quad \mathbf{s}_i = \mathbf{A}\mathbf{p}_i, \quad \mathbf{q}_i = \mathbf{G}_{i-1}\mathbf{s}_i, \tag{49a}$$

$$d_i = \mathbf{p}_i^{\mathrm{T}}\mathbf{s}_i, \quad \gamma_i = \mathbf{q}_i^{\mathrm{T}}\mathbf{s}_i, \quad \beta_i = \gamma_i / d_i, \tag{49b}$$

$$\mathbf{G}_i = \mathbf{G}_{i-1} - (\mathbf{p}_i\mathbf{q}_i^{\mathrm{T}} + \mathbf{q}_i\mathbf{p}_i^{\mathrm{T}})/d_i + (1+\beta_i)\mathbf{p}_i\mathbf{p}_i^{\mathrm{T}}/d_i. \tag{49c}$$

For each i, the matrix $\mathbf{G}_i$ is a positive definite symmetric matrix, which is also given by the formula

$$\mathbf{G}_i = \mathbf{C}_i\mathbf{C}_i^{\mathrm{T}} + \mathbf{B}_i, \tag{50}$$

where $\mathbf{B}_i$ and $\mathbf{C}_i$ are the matrices (45) and (47b) respectively. Because $\mathbf{C}_n = 0$ and $\mathbf{B}_n = \mathbf{A}^{-1}$, we have $\mathbf{G}_n = \mathbf{A}^{-1}$. The solution of $\mathbf{A}\mathbf{x} = \mathbf{k}$ is therefore $\mathbf{z} = \mathbf{G}_n\mathbf{k}$. Algorithm (49), with suitable modifications, is the basis for the variable metric routines that have been developed for nonquadratic as well as quadratic optimization.

Conjugate direction routines and, in particular, the conjugate gradient routine have many interesting and useful properties. For example, in the conjugate gradient algorithm (27), the function $f(\mathbf{x})$ and the error function $|\mathbf{x} - \mathbf{z}|$ are diminished at each step. The conjugate gradient routine (27) can terminate in fewer than n steps. This occurs when $\mathbf{k}$ is orthogonal to some of the eigenvectors of $\mathbf{A}$. It also occurs when $\mathbf{A}$ has fewer than n distinct eigenvalues, unless round-off errors intervene. If the eigenvalues of $\mathbf{A}$ are clustered about m values, the vector $\mathbf{x}_{m+1}$ will be a good estimate of the solution $\mathbf{z}$ of $\mathbf{A}\mathbf{x} = \mathbf{k}$. This means that often a good estimate of $\mathbf{z}$ is obtained in fewer than n steps. In 1951, in a physical application, Steifel and Hochstrasser obtained a satisfactory estimate of $\mathbf{z}$ in 90 steps with $n = 106$ [94]. Numerical experiments were carried out at INA by Hayes, Hochstrasser, Stein, Wilson, and others. Ill-conditioned as well as well-conditioned matrices were used, including the Hilbert matrices. Singular matrices were also studied. We considered nonsymmetric as well as symmetric matrices, using an appropriate conjugate gradient algorithm in each case. In all cases we obtained excellent results when high precision arithmetic was used. Because of machine limitations, we restricted ourselves to relatively small ($n \leq 12$) matrices. As a result we did not make an in-depth study of round-off errors. Because of round-off errors, the conjugate gradient routine frequently does not yield the exact solution in n steps. If a satisfactory estimate of the solution is not obtained after n, $n+1$, or $n+2$ steps, the algorithm should be restarted with the last estimate of the solution as the new initial point. Alternatively, we can use the methods proposed by Lanczos for dealing with round-off errors. Some have used a preconditioning scheme on the matrix A. Many of these are equivalent to using Algorithm (36) with a wise choice of H and K. Others have combined the conjugate gradient method with other routines to obtain effective means for obtaining solutions of systems of linear equations arising in the study of linear (and nonlinear) partial differential equations. At present, extensions of the conjugate gradient method play an important role in numerical applications of nonquadratic as well as quadratic optimization.

An interesting application of conjugate gradients was made by Dr. David Sayre of IBM. He sought to find a least squares solution of a system of more than 10,000 nonlinear equations in about 5,000 unknowns. His method was to linearize the equations, find a least squares estimate of the linearized equations, use the estimate to obtain a new linearization, and repeat the operation. To obtain his least square estimate of the linearized equations, he used only five steps of a conjugate gradient algorithm of the type (36). The results are described in the following excerpt.

10,000 Equations and 15 Hours Later

It usually takes microseconds—perhaps as long as several seconds—for the computer to solve most problems. However, Dr. David Sayre of IBM waited almost 15 hours as a System/360 Model 91 computed the answer to his particular problem—a problem a team of scientists once worked two years to complete.

A mathematician at the T. J. Watson Research Center, Sayre has been engaged in refining the structure of a protein called rubredoxin. Beginning with an X-ray map at a resolution of 2.5 angstroms, Sayre used the Model 91 to process a complex system of more than 10,000 non-linear equations—one of the largest equation systems ever programmed for computer solution. A critical technique he employed, called conjugate gradients, was developed by Professor Magnus Hestenes of UCLA and was adopted by Sayre at the suggestion of a research center colleague, Dr. Philip Wolfe.

With the computation completed, and the resolution mathematically refined to 1.5 angstroms—about the distance between the centers of neighboring atoms—Sayre was able to identify some 400 of the protein's 424 non-hydrogen atoms.

In significantly reducing the computation time required for such high-resolution studies, Sayre's work may lead to improved understanding of molecules like DNA and RNA, key elements in the reproduction of human cells.

Dr. Sayre examines the structure of the protein, rubredoxin, on this electron-density map.

For an account of the later history of the conjugate gradient and Lanczos algorithms see [133].

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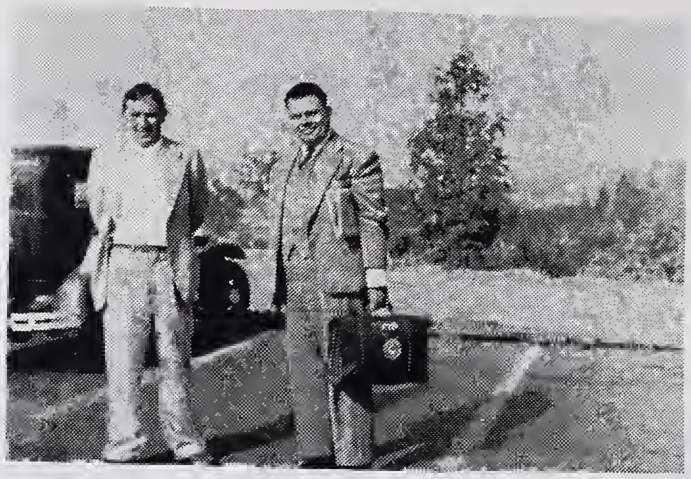
PHOTOGRAPHS AND CARTOONS

Edward U. Condon, Fourth Director NBS 1945–1951

John H. Curtiss, First Chief, Applied Mathematics Division 1947–1955

NBS-INA, Building 3U on UCLA Campus

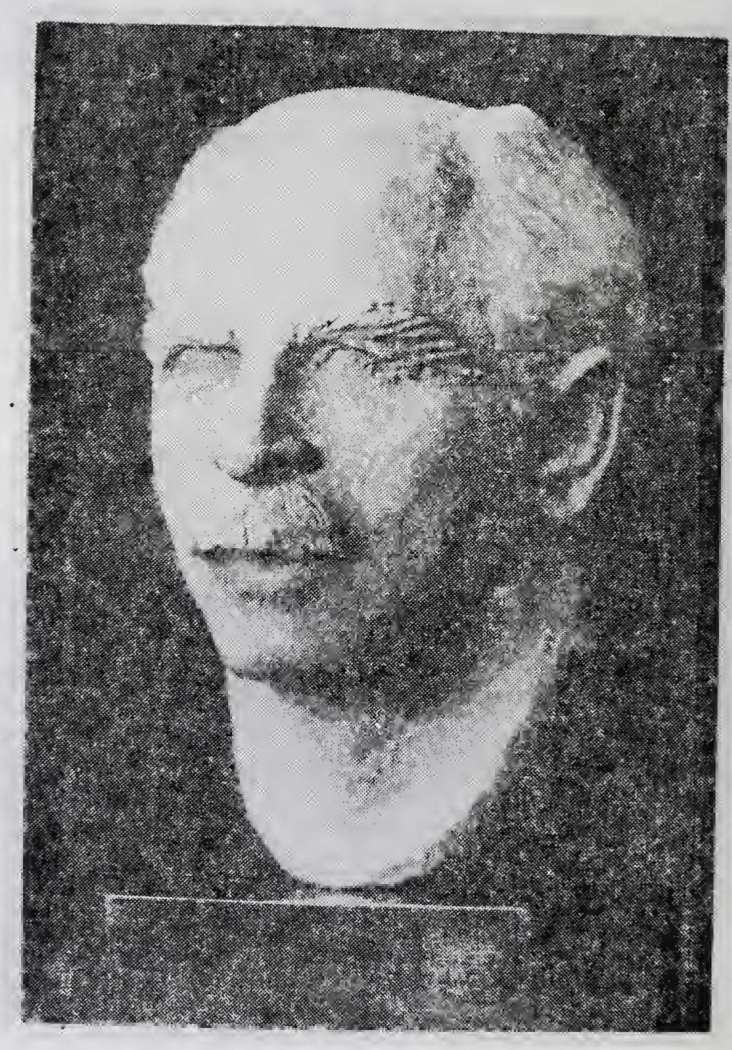
Mina Rees, John H. Curtiss, and Olga Todd



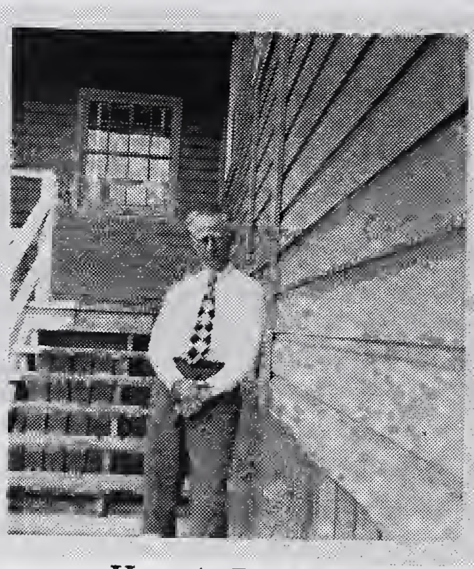
John Todd and Edward U. Condon (in the INA parking lot)

Albert S. Cahn, Jr.

Gertrude Blanch



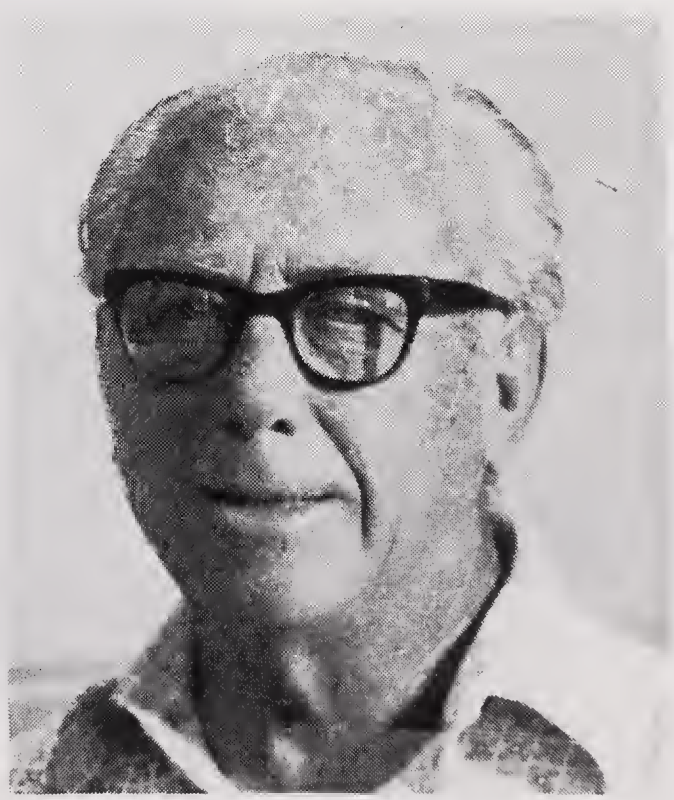
Otto Szász (from his Collected Papers)



Hans A. Rademacher

Douglas R. Hartree, Acting Director INA 1948 (using a Brunsvign B20 hand computer) [From obituary notice by C. S. Darwin, Biographical Notices of Fellows of the Royal Society 4 (1958), p. 102–116.]

Roselyn Siegel Lipkin



Edwin F. Beckenbach, UCLA Liason Officer

Unknown, George E. Forsythe, and William E. Milne

Robert E. Greenwood and Robert H. Cameron

Olga Todd

John H. Curtiss, Acting Director INA 1948–1949

Tobias Dantzig and Alexander M. Ostrowski (Tobias Dantzig's son, George B. Dantzig, who created linear programming, was closely associated with NBS and in particular with relevant parts of INA activity.

George E. Forsythe, John Todd, and John W. Green

J. Barkley Rosser, Director INA 1949–1950

Group including: Mark Kac, Edward J. McShane, J. Barkley Rosser, Aryeh Dvoretzky, George G. Forsythe, Olga Todd, Wolfgang R. Wasow, and Magnus R. Hestenes

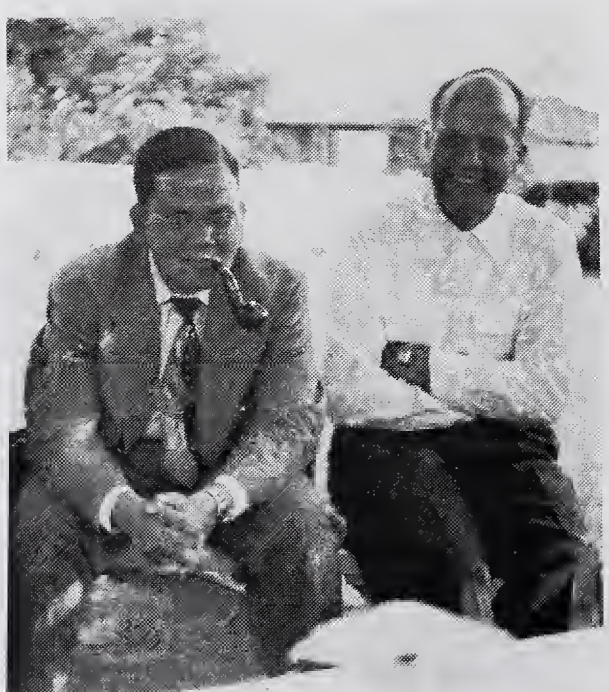
Mrs. and Mr. Theodore S. Motzkin

Wolfgang R. Wasow and Magnus R. Hestenes

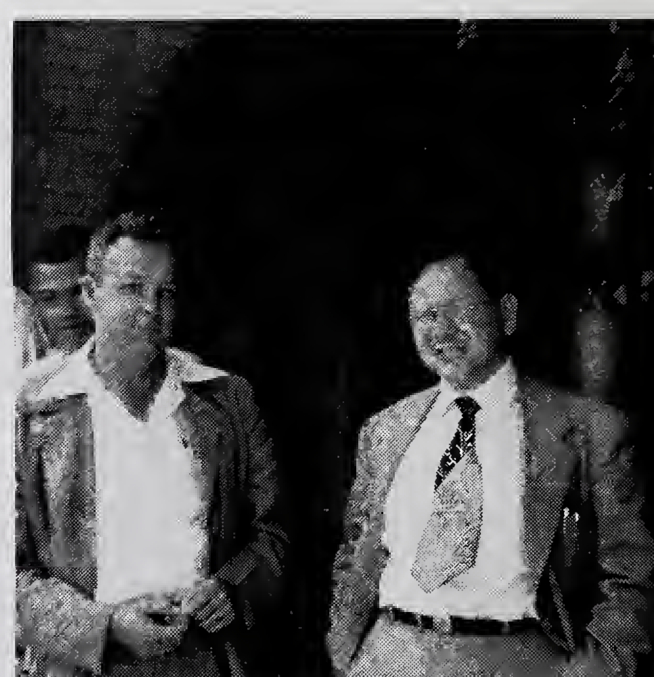
Cornelius Lanczos and Mrs. Arnold D. Hestenes

Eduard Stiefel [from ZAMP 30 (1979)]

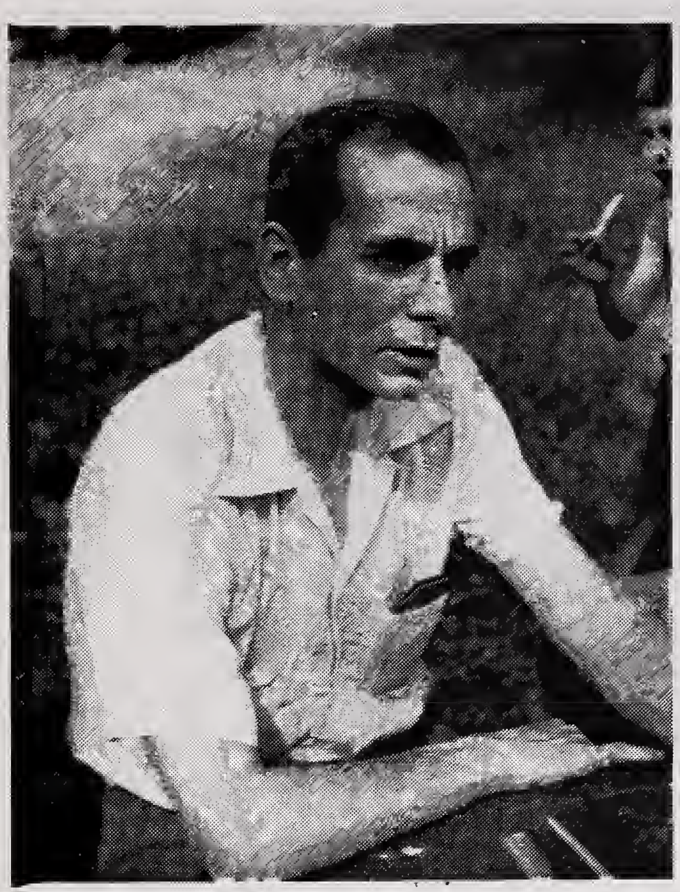
George E. Forsythe and Isaac J. Schoenberg



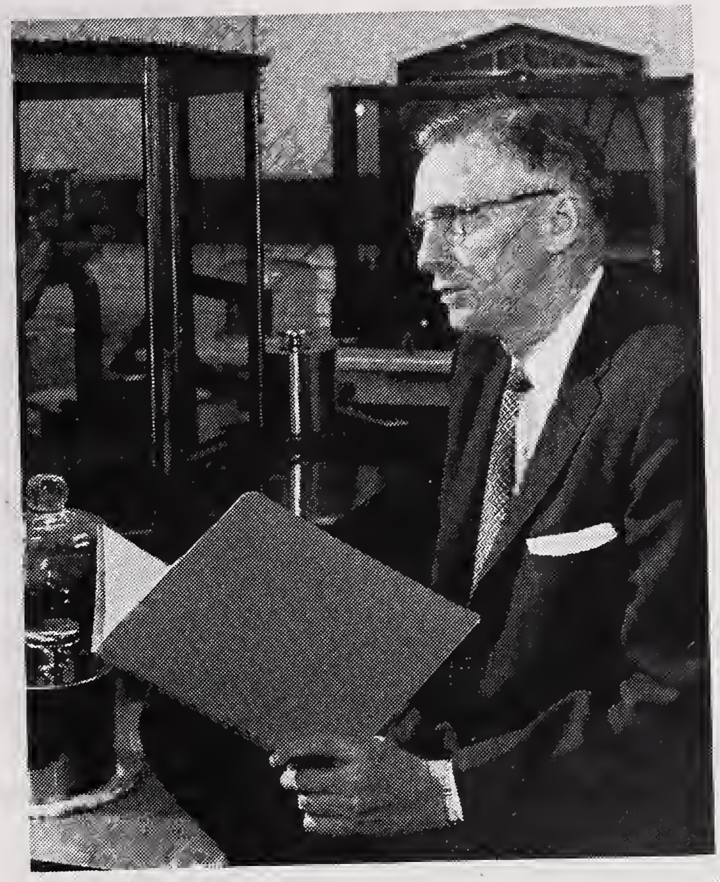
Everett C. Yowell and Thomas H. Southard



Arnold D. Hestenes and Everett C. Yowell



Fritz John, Director NBS-INA 1950–1951

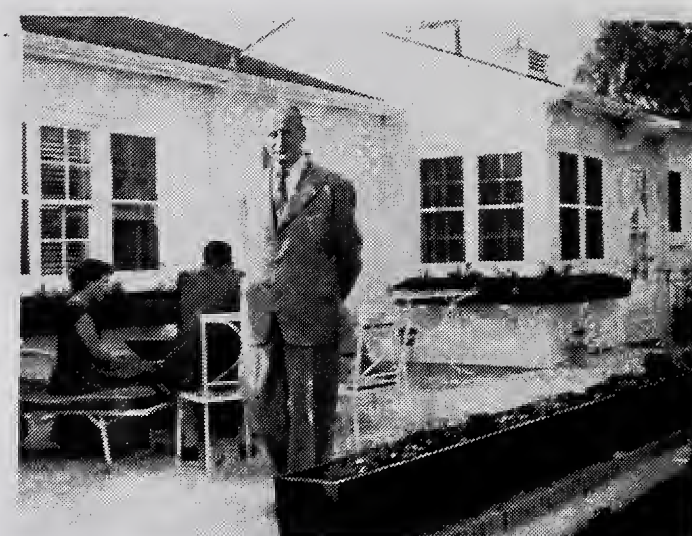


Allen V. Astin, Fifth Director NBS 1951–1969

Isaac J. Schoenberg

Charles B. Tompkins, Director NBS-INA 1953–1954

Derrick H. Lehmer, Director NBS-INA 1951–1953 with Mrs. Lehmer (left) and Mrs. Arnold D. Hestenes (right)



J. G. van der Corput

"Try It Again, Men, And Be Sure You Get This Answer"

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"Yes, I'm Getting Quite A Charge Out Of It"

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APPENDIX A

JOHN HAMILTON CURTISS 1909–77

We are greatly indebted to John Curtiss for his leadership and foresight in fostering the development of mathematics pertinent to machine computation. He had the ability to interest prominent mathematicians to join him in this new development. The present book describes some of the mathematicians whom he enticed to devote their time and efforts in this direction. It is most appropriate that we include in this book the story of John Curtiss himself. Fortunately, such a story has already been written by John Todd. We present this article now as our story of his life.

John Hamilton Curtiss, 1909–1977

JOHN TODD

John Hamilton Curtiss was chief of the Applied Mathematics Division of the National Bureau of Standards from 1946 to 1953. He was largely responsible for the planning and construction of SEAC and SWAC and for the procurement of the first UNIVACs for federal establishments.

Keywords and phrases: automatic computer, SEAC, SWAC, National Bureau of Standards, John Curtiss, obituary
CR category: 1.2

1 Vita

John Hamilton Curtiss was a man of many talents: first and always a mathematician, but also a highly able administrator, musician, and tennis player. During the years 1946–1953 he was at the National Bureau of Standards (NBS) and played a vital role in the development, procurement, and widespread application of computers in the United States. I was with him at NBS in 1947–1948 and 1949–1953, and I shall discuss his contributions to the computer field as I remember them.

John Curtiss was born on December 23,[1] 1909, into an academic environment. His father, D. R. Curtiss (1878–1953), was professor of mathematics at Northwestern University, was president of the Mathematical Association of America in 1935–1936, and wrote a standard introduction to complex variable theory (1926), still in print. His uncle, Ralph H. Curtiss, was professor of astronomy at the University of Michigan.

After graduating with highest honors from Northwestern University in 1930, John Curtiss obtained an M.S. degree in statistics under H. L. Rietz at the University of Iowa, then one of the leading centers in the Midwest for mathematical training. Two of Curtiss's fellow graduate students there were S. S. Wilks and Deane Montgomery, who went on to distinguished mathematical careers at Princeton University and the Institute for Advanced Study, respectively.

John Curtiss then went to Harvard University, where he earned his Ph.D. in 1935 under Professor J. L. Walsh. His first job after obtaining the doctorate was an instructorship in mathematics at Johns Hopkins University in 1935–1936. In 1936 he joined the mathematics faculty at Cornell University, where he taught until entering the U.S. Navy in January 1943. He was stationed in Washington, DC with the quality control section of the Bureau of Ships until April 1946, when he was discharged with the rank of Lt. Commander.

He immediately joined the NBS as an assistant to the director, E. U. Condon, and was initially responsible for statistical matters. On July 1, 1947, he was appointed chief of a new division of the NBS initially called the National Applied Mathematics Laboratories; later the designation Applied Mathematics Division (AMD) was used and I shall keep to this. John Curtiss remained at the NBS until mid-1953, except for a semester as visiting lecturer at Harvard University in 1952. He spent a year at the Courant Institute of New York University, and was executive director of the American Mathematical Society (AMS) in Providence, RI from 1954 to 1959. In 1959 he became professor of mathematics at the University of Miami, Coral Gables, where he worked intensively on one of his first areas of interest: approximation theory in the complex domain. One by-product of this period was a graduate text on complex variable theory (1978). He died of heart failure at Port Angeles, WA on August 13, 1977, while en route to the AMS summer meeting in Seattle.

While I cannot recall ever seeing John Curtiss at the console of a computer—he always said that I was involved in the salt mines of computing—his interest in numerical analysis was considerable. He wrote one paper on numerical algebra (1954*b*) and edited the proceedings of an important symposium (1956). Although the main body of his work on approximation theory is peripheral to practical computation, as a statistician he was deeply interested in "Monte Carlo." One paper on this subject (1950), delivered at an IBM confer-

Author's address: Department of Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125.

[1] December 23, incidentally, is of great importance in the history of mathematics. R. Fricke wrote in the *Encyklopädie d. Math. Wiss.* (II, B3, p. 183) that Jacobi said it was the birthday of elliptic functions. Neither B. C. Carlson nor I has been able to check this statement, but Euler reported on the fundamental work of Fagnano to the Berlin Academy on this date in 1751, and it was on this date in 1799 that Gauss completed the proof of the expression for the arithmetic-geometric mean in terms of an elliptic integral.

ence in 1949, was much acclaimed, and another (1954*a*) was translated into Russian. He gave courses on numerical algebra at NYU in 1953–1954 and on numerical analysis at the University of Miami. During his time in Providence he made a careful analysis of the book-sales policies of the AMS.

John Curtiss was quite sure that the nascent computing fraternity had to become a national society with publications of its own if it were to develop appropriately and be able to exert influence. Thus he helped enthusiastically in the organization of the Eastern Association for Computing Machinery (ACM), which dropped the regional adjective from its title in 1948. He was the first president in 1947 and always encouraged his staff to participate in the work of this and other professional organizations. Of these, Franz L. Alt, Harry D. Huskey, and George E. Forsythe later became presidents of ACM and Thomas H. Southard was president of SIAM. Curtiss also saw that publications were supported, in particular *Mathematical Tables and Other Aids to Computation* and the *Pacific Journal of Mathematics*.

I will restrict myself fairly severely to the time period 1946–1953 and to activities involving machines. But it is appropriate to point out here that John Curtiss's contribution to the development of modern numerical mathematics cannot be overestimated. He realized that any experienced pure mathematician could find attractive, challenging, and important problems in numerical mathematics, if the person chose to do so. From the death of Gauss in 1855 to 1947 the field of numerical mathematics was, with a few exceptions, cultivated by nonprofessional mathematicians whose real interests lay elsewhere. Accordingly Curtiss recruited professionals from far and near to take part in the programs he had envisaged. Indeed, nearly all of his recruits contributed significantly to his program.[2]

John Curtiss once documented the remarkable success of the operation he planned by counting the papers presented at the 1952 International Congress of Mathematicians by various organizations. NBS was in the middle of the top seven; the others were the University of California at Berkeley, the University of Michigan, the University of Chicago, the Institute for Advanced Study, Harvard University, and the University of Pennsylvania—all organizations founded long before the Applied Mathematics Division of NBS.


John Curtiss in 1976.

For further details of the numerical mathematics programs at the NBS see Lowan (1949), Blanch and Rhodes (1974), and Todd (1975).

2 "The Prospectus"

NBS was no stranger to computing equipment. It had been responsible for the Mathematical Tables Project of the Works Progress Administration in New York since 1938. This group was supported during World War II by the Applied Mathematics Panel of the Office of Scientific Research Development and from 1946 by the organization that later developed into the Office of Naval Research (ONR). Led by Arnold N. Lowan, the group included Milton Abramowitz (later chief of the Computation Laboratory), Ida Rhodes, Gertrude Blanch, Herbert E. Salzer, and Irene A. Stegun.

While John Curtiss was first concerned with statistical matters within NBS, he soon had a national responsibility. Several incidents led to this. In 1945, Eckert and Mauchly, who were largely responsible for the ENIAC, approached the Census Bureau (which, like NBS, was part of the U.S. Department of Commerce) with the suggestion that a computer could facilitate its work—in the coming 1950 census, for example. This suggestion was discussed by the Science Committee of the Department of Commerce, which asked NBS for technical advice. The final agreement (April 1946) was that the Census Bureau would transfer funds to NBS, which would select a suitable computer and purchase it. The Army Ordnance

[2] Among these was J. C. P. Miller, from the Cambridge University Mathematical Laboratory. Happily, the adjective *late* used in a recent issue of the *Annals* (Volume 1, Number 2, October 1979, p. 99) does not apply. Miller continues to make distinguished contributions to numerical mathematics.

Department also transferred funds to the Electronics Division of NBS for the development of computer components.

At this time Condon instructed Curtiss to survey the federal needs for computers and for a national computing center. This investigation had its source in ONR, and Rear Admiral H. G. Bowen suggested that NBS and ONR should jointly establish such a center, to develop as well as use computers. Funds for this purpose were transferred in September 1946. In other countries, similar plans were being considered (see Todd 1975, p. 362).

Curtiss's investigations led to a broadening of the program. He realized very early, for example, that the mathematics needed to exploit the new

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John Todd is currently Professor of Mathematics at the California Institute of Technology. He was attached to the National Bureau of Standards in 1947–1948 and 1949–1957. His interest in computing matters began during World War II when he was a Scientific Officer in the British Admiralty. He has written extensively on numerical mathematics.

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computers had also to be developed, an opinion shared by Mina Rees of ONR (1977). The program he formulated was described in the prospectus issued in February 1947. The AMD was to have four sections:

1. Institute for Numerical Analysis (INA)—to be a field station at UCLA (the University of California at Los Angeles).
2. Computation Laboratory (CL).
3. Statistical Engineering Laboratory (SEL).
4. Machine Development Laboratory (MDL).

The last three were to be in Washington, DC. The nucleus of the CL was to be the Lowan group. The program of the AMD to be guided, within NBS operations, by an Applied Mathematics Executive Council consisting of representatives of various federal agencies and some outside experts. Later the title of this group was changed to Applied Mathematics Advisory Council (AMAC). A total staff of about 100 was contemplated and of that only about 30 were on the NBS payroll when the AMD was founded.

The AMD came into being on July 1, 1947. The prospectus had been a remarkable document insofar as there was little need to change its contents as time passed. Curtiss undertook a massive recruitment program to implement the aims of the AMD. He was highly qualified for this activity, with his outgoing personality and many academic contacts. Fortunately, too, the time was opportune for such an expansion, because many mathematicians were being demobilized from their World War II activities—a number of them fresh from some experience with applied mathematics.

The soundness of the original structure of AMD is clear from the fact that despite a succession of NBS directors and several reorganizations, the present (1979) organization is essentially unchanged, apart from the transfer of the MDL activities elsewhere. The original division has been elevated to a center and now has four divisions: Mathematical Analysis, Operations Research, Scientific Computing, and Statistical Engineering.

Curtiss supported his staff with an administrative and intellectual climate conducive to scientific development. In this context let me mention an instance of his loyalty to his staff. On one occasion, I was unexpectedly asked to defend an awkward personnel action, and my superiors took advantage of Curtiss's absence at INA to "chew me up." I complained to Curtiss by telephone, and he immediately composed a seven-page memorandum to those concerned, essentially a paper on Classification and Recruitment Problems Peculiar to a Mathematical Research Unit. Beginning with the assumption that the federal government wanted a mathematics activity, he explained how to operate it economically and efficiently. In my remaining years at NBS, I had no further problems in this connection.

The years I am discussing were not the happiest in Washington and some of our staff were involved in loyalty and security problems. John Curtiss saw to it that they got the best legal advice.

3 Procurement Problems

By late 1946, once certain legal situations were resolved, NBS had funds available for two computers. The first was the UNIVAC for the Census Bureau, contracted for with the Eckert-Mauchly organization in 1946, and the second was the NBS computer (financed by ONR), contracted for with the Raytheon Company early in 1947.

The terms of these contracts were discussed by various committees, by advisors (notably, G. Stibitz), and by the AMAC. There were many complications, both administrative and technical. During these discussions the one UNIVAC became three; one for the Air Comptroller and another for the Army Map Service were added.

It is appropriate to mention here the division of responsibilities between the Applied Mathematics Division and the Electronics Division of NBS. The AMD was responsible for the logic design of the computers and their suitability for the jobs envisaged, and for initial liaison with the contractors. The Electronics Division was responsible for the soundness of the design of components and for all engineering matters. Once the development was complete, the divisions were to share the liaison and documentation duties.

The chief of the MDL from 1946 was E. W. Cannon, who succeeded John Curtiss as chief of the AMD in 1953. Ida Rhodes, originally with the Mathematics Tables Project, was active in ensuring the suitability of proposed designs and later in educating the coders and programmers.

Early in 1948, as it became clear that none of these machines would be completed on schedule, two enormously significant events took place. The Air Comptroller, while awaiting the delivery of the UNIVAC, realized that a small "interim" computer to be developed at NBS would provide useful experience. This led to SEAC, discussed in Section 4 below. The Air Materiel Command wanted two computers, one for Wright Field and one for INA, but no supplier could be found. Consequently, it accepted a proposal for a modest machine to be developed at INA by Harry D. Huskey. This led to SWAC; see Section 5.

To end this historical sketch, the Census UNIVAC was completed early in 1951 and was dedicated on June 16, 1951. The UNIVAC for the Air Comptroller was completed in February 1952, and the one for the Army Map Service was completed in April 1952. The Raytheon machine for NBS was never completed, but a related machine was delivered to the Naval Air Missile Test Center at Point Mugu, CA in 1952.

4 SEAC

The NBS Interim Computer, later called SEAC (Standards Eastern Automatic Computer), was constructed for the Air Comptroller by a group in the NBS Electronics Division (led by S. N. Alexander) beginning in the fall of 1948. The MDL collaborated in the design, and it was agreed that as soon as the computer became operational it would be moved to the CL. At that time I was the chief of the CL, and we had a considerable group (led by Alan J. Hoffman) working for the Air Comptroller on linear programming, a subject just being developed by G. B. Dantzig and his associates.

In about 15 months SEAC became productive. On April 7, 1950, my hand held by R. J. Slutz, I ran my first program: solving the Diophantine equation $ax + by = 1$. Actually a and b were originally taken to be the largest pair of consecutive Fibonacci numbers that fitted into the machine ($<2^{44}$); this was chosen to give the slowest Euclidean algorithm. The day before, Franz Alt had run a factorization program using a small sieve.

SEAC was dedicated on June 20, 1950. Originally it had a 512-word delay-line memory, but 512 words of electrostatic memory were added. The original teletype input/output was supplemented by magnetic wire. For more technical information, see NBS (1947, 1950, 1951, 1955), Greenwald et al. (1953), Shupe and Kirsch (1953), and Leiner et al. (1954).

It was originally quite a distance from the CL to SEAC's building, and my coders/operators had to cross and recross the NBS grounds to use the computer. On a visit to Los Alamos in 1951, after I had described (perhaps too enthusiastically) the current state of our operations, the laboratory authorities there decided that SEAC was just the thing they needed for their weapon-related computations. Accordingly they preempted SEAC, providing their own crew (for security reasons and for educating them in the use of computers). Even less time was then available for developmental work, and pleas to move the machine to the CL, where we now realized how odd minutes could be used effectively, were rejected on the grounds that the delicate equipment might not survive the trip. John Curtiss finally negotiated with the AEC for the construction of a cinder-block building abutting the SEAC building, and those of us in direct contact with the machine moved into the new structure. A few years later the machine was moved to the CL, where it operated until it was retired on April 23, 1964.

5 SWAC

In Section 3 I noted the origin of the Air Materiel Command machine, later called SWAC (Standards Western Automatic Computer). This project began from scratch in January 1949, and the first Williams tube machine to be completed in the United States was dedicated on April 7, 1950. Just as the British ACE was designed by a mathematician (A. M. Turing), the SWAC was designed by Harry D. Huskey, who was trained as a mathematician. It was built among and for mathematicians.

There was a rather long period of debugging, but in due course all troubles were overcome and

SWAC became a reliable machine with many significant accomplishments. When the INA operation closed in 1954, SWAC was transferred to UCLA and remained in operation until 1967. See the accompanying article about SWAC by Huskey; also see NBS (1947, 1950, 1951, 1955), Huskey (1950), and Huskey et al. (1953).

6 Conclusion

John Curtiss, with the full support of NBS director E. U. Condon, and with modest encouragement from various federal agencies, accelerated the progress of the United States toward a preeminent position in the construction of computers and their exploitation for scientific computations. I have already mentioned the use of SEAC by the AEC; rocket and comet orbits were computed on commercial equipment at INA (Herrick 1973), and perhaps the first automatically computed earth-moon trajectory was done on SEAC (Fröberg and Goldstein 1952).

Had Curtiss been able to stay at NBS and had support for INA been continued,[3] there is no doubt that the mathematical development would have kept up with the enormous achievements of the engineers. We who remained did what we could to carry on the work for which Curtiss laid solid foundations. Those of us who were with him at NBS enjoy getting together and recalling the exciting times of 1946–1953 and are grateful to have had the privilege of working with a fine American mathematician.

7 Postscript—Some Personal Reminiscences

My own first contact with John Curtiss was a letter dated early in 1947, enclosing the prospectus and inviting me to consider joining INA. During World War II, I had been active in organizing an Admiralty Computing Service in Great Britain. Later, with my colleagues A. Erdélyi and D. H. Sadler, I suggested the formation of a National Mathematical Laboratory, later established as a division of the National Physical Laboratory. During that time my wife, Olga Taussky, and I had many contacts with American mathematicians stationed in or visiting Europe, especially H. P. Robertson, H. M. MacNeille, G. Baley Price, R. Courant, and J. von Neumann. They were aware of my activities, and I was in correspondence with members of the Applied Mathematics Panel. Some of these people probably suggested my name to John Curtiss. We arrived in New York on a troop ship late in September 1947. Our first contact with the computer world outside Washington was at the Aberdeen Meeting of ACM on December 11–12, 1947.

John Curtiss was a bachelor who enjoyed fast cars and plenty of good food and drink. In introducing us to Washington society he asked me to arrange a sherry party in his apartment. I provided sherries of varying quality and served them according to his evaluation of the guests, reserving the Bristol Cream for the director. For the benefit of the many visitors to INA he compiled a list of restaurants labeled according to the civil service gradings P1 to P8.

He did not find the Civil Service regime too convenient, and much of his activity was spent maintaining contacts with other agencies, often after regular hours. He dictated a diary late at night; a transcription was circulated to his staff the next day so that we were aware of what commitments he had made.

He was not happy on planes and did not travel to Europe until 1976. A letter from him dated May 11, 1976, from the Mathematical Research Institute at Oberwolfach is addressed to me as "The Savior of Oberwolfach" (As British naval officers, G. E. H. Reuter and I were able, in 1945, to prevent the dissolution of an institution that has since made great contributions to mathematics, including formal languages, complexity theory, many aspects of numerical analysis, and, for instance, computerized tomography.) He complained about staying in "magnificent old fire traps" and characterized one of the famous Lon-

[3] In 1952 there was a change of administration and the NBS was involved in the ADX2 (battery additive) controversy. Government scientists were reprimanded for not taking account of the play of the marketplace. The NBS director was asked to resign, but was reinstated, and the NBS decisions were fully supported by a special committee appointed by the National Academy of Science. One consequence of this was an investigation of the NBS as a whole by an ad hoc committee chaired by M. J. Kelly. In the report of this committee (submitted to the Secretary of Commerce on October 15, 1953) the AMD was one of two divisions that were specially commended. The whole of the AMD was at that time in a precarious financial situation due in part to a decision of the Department of Defense not to permit transfers of funds to NBS—and the AMD had relied very heavily on such transfers from its beginning. The University of California was not in a position to take over the operation of INA, but it kept the SWAC and the valuable library and continued a numerical analysis research project on a modest scale. Private industry and universities benefited from the well-trained staff they were able to recruit from the closing of the INA operation and heavy reductions in t' Washington workforce.

don clubs as "the awfullest fire trap of all, but interesting." He indicated two remembrances of England, the first "an infinite series of near-head-on-collisions," and the second musical: "I recently got a record of Elgar's organ music including the Sonata in C played on the organ at Colston Hall, Bristol, which we inspected amid chaotic preparations for a Salvation Army Choral concert. Then we heard the Sonata itself, included by coincidence in a noon recital in Hertford Chapel in Oxford (and played too slowly)."

My last meeting with John Curtiss was at the 1976 Los Alamos Research Conference on the History of Computing. He said then that he thought that historians, so far, had not fully appreciated the contribution of the National Bureau of Standards in the field. I hope this essay will begin to put things in balance.

8 Acknowledgments

I am indebted to John Curtiss's sister Alice (Mrs. Edwin F. Beckenbach) for much personal information. I am also grateful to Nancy Stern for careful criticism of a draft of this paper and to Churchill Eisenhard and Henry Tropp for comments on a second version.

I have had access to a five-year progress report Curtiss wrote in 1953. All the material I have used, including a file of the periodical reports of the AMD (1947-—), will be deposited at an appropriate center.

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APPENDIX B

TWO PAPERS BY JOHN H. CURTISS

The first of the following two papers is one of the early papers on the Monte Carlo method and had a great influence in the development of this method. Though it was never formally published in a scientific journal, the paper was presented at the Seminar on Scientific Computation, November 1949, IBM Corporation. Because of its significance we are pleased to present this contribution at this time. The second paper is a report on the progress of the Institute for Numerical Analysis 1947–51. It contains a bibliography of the work accomplished during the first 3 years. Attention is, in particular, invited to the concluding paragraphs.

Sampling Methods Applied to Differential and Difference Equations

JOHN H. CURTISS

National Bureau of Standards

THE PURPOSE of this paper is to present, as simply as possible, a part of the theory underlying the use of probability sampling techniques in the numerical solution of boundary value problems. The sampling procedures under discussion consist of repeated trials, or realizations, of a type of stochastic process which physicists call a random walk (or random flight) with absorbing barriers. This type of process has recently become familiar to statisticians in connection with sequential analysis.

In practice, the trials are actually to be carried out by computing machines or hand computers. However, the problem of instructing machines or hand computers as to how best to execute and check the computations will be touched on only briefly in this paper. Another limitation on scope is that eigenvalue problems will be avoided by considering equations in which only the derivatives or differences of the unknown function appear, not the function itself. Still another limitation is that the basic asymptotic theorems are only sketched.

An attempt has been made to state, with reasonable generality, a number of facts which have heretofore been presented only for special cases, or which have never been formally published, although they may be common knowledge among specialists. Some of the material in Sections 11 to 17 inclusive appears to be new. An effort has been made to compile a bibliography which is extensive enough to serve at least as an introduction to the literature bearing upon the connection between differential equations (particularly of the type to be considered here) and stochastic processes.

This introduction would not be complete without an acknowledgement of the author's debt to his colleagues in the National Bureau of Standards; in particular, to Dr. W. Wasow and Dr. R. P. Peterson, Jr., with whom a number of conversations on the subject matter were held. Dr. Wasow and Dr. Forman Acton read a first-draft of the manuscript and made many helpful suggestions. The development in Section 17 is almost entirely due to Dr. Wasow, and part of the exposition in Section 13 closely follows a discussion which he transmitted to the author in a letter. The author is grateful to Dr. Wasow for permission to include this material.

The article undoubtedly shows traces of the influence of Prof. W. Feller, who conducted seminars at the Institute for Numerical Analysis of the National Bureau of Standards in the summer of 1949. The manuscript of Dr. Feller's forthcoming book on probability, which will probably become a standard reference, became available at the Institute after the present paper was largely completed. Unfortunately, limitations of time schedule made it impossible to consult this manuscript in the course of the preparation of this article, although the paper would probably have thereby benefited considerably.

Historical and Bibliographical Notes 2

It has lately become fashionable to apply a rather picturesque name, the "Monte Carlo Method", to any procedure which involves the use of sampling devices based on probabilities to approximate the solution of mathematical or physical problems. The Monte Carlo Method has, so far, largely been used in connection with functional equations and quadratures. No comprehensive technical exposition of the method has as yet appeared. A stimulating philosophical introduction to the Monte Carlo Method has been given by Metropolis and Ulam[18], and a number of interlaboratory reports have appeared. In such publications, and in word of mouth transmittal of computing techniques, it is natural to omit references to any existing background literature. Consequently, misconceptions regarding the novelty of the method have arisen from time to time among persons interested in the applications rather than the history.

The use of functional equations in connection with probability problems arising in games of chance dates back to the earliest beginnings of probability theory, and was familiar to de Moivre, Lagrange, and Laplace. The connection between probabilities and the differential equations of mathematical physics is also an old story to theoretical physicists. During the early years of the present century, it was studied particularly extensively by Einstein, Smoluchowski, Lord Rayleigh, Langevin, and many others. Good bibliographies will be found at the end of References 3, 24 and 30. More specifically, in 1899 Lord Rayleigh[22] proved that a random walk in one dimension (of the type considered below in Section 4 but without absorbing barriers) yields an approximate solution of a parabolic differential equation. The relationship between random flights (i. e., random walks in more than one dimension) and the first boundary value problem, or Dirichlet problem, for elliptic difference and differential equations was established by Courant, Friedrichs, and Lewy in 1928[5]. Mathematical interest was stimulated by the publication in 1931 of a celebrated paper by Kolmogorov[14] dealing with the relationship between stochastic processes of the Markoff type and certain integro-differential equations. Shortly afterwards, the well-known book of Khintchine appeared[13], followed by papers by Feller (for example, Ref. 9) and others.

This is, of course, not intended to be an exhaustive survey. Enough has been said to indicate that the theory of the Monte Carlo Method has a rather distinguished pedigree. The novelty which the method possesses lies chiefly in its point of view. With few exceptions, the authors cited above proceed from a problem in probabilities to a problem in functional equations, whose solution is then obtained, or at least proved to exist, by classical methods and furnishes the answer to the probability problem. In the Monte Carlo Method, the situation is reversed. The probability problem (whose solution can always be approximated by repeated trials) is regarded as the tool for the numerical analysis of a functional equation. Or alternatively, in a physical problem which, classically, would call for an analytic model, the equivalent probability problem is regarded as an adequate model in itself, and derivation of an analytic equivalent is considered to be superfluous. The use of probability theory in this way to solve physical problems was apparently first suggested by von Neumann and Ulam.

But it is worth noting that the Monte Carlo Method is not at all novel to statisticians, as was pointed out by Prof. John Wishart at a symposium on the Monte Carlo Method held in 1949 at Los Angeles, Calif.[19]. For more than fifty years, when statisticians have been confronted with a difficult problem in distribution theory, they have resorted to what they have sometimes called "model sampling". This process consists of setting up some sort of urn model or system, or drawings from a table of random numbers, whereby the statistic whose distribution is sought can be observed over and over again and the distribution estimated empirically. The theoretical distribution in question is usually a multiple integral over a peculiar region in many dimensions, so, in such cases, "model sampling" is clearly a Monte Carlo Method of numerical quadrature. In fact, the distribution of "Student's t" was first determined in this way. Many other examples can be found by leafing through the pages of Biometrika and the other statistical journals[a].

Most of the technical literature referred to above on the relation between probabilities and functional equations deals with equations which can loosely be described as of parabolic type. The amount of published material on the elliptic case is relatively meager, and for that reason this case has been chosen as the central topic of this paper. The basic references at the moment seem to be 5, 13, 16, 20, 23, and the ever-growing literature on sequential analysis, summarized to 1946 by Wald[28, b]. Several other references will be given as the exposition proceeds.

A General Remark 3

It was observed by Courant, Friedrichs, and

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a. It should also be mentioned that the computation of important constants such as π by statistical sampling methods is an old trick mentioned in many textbooks. An interesting computation of π of this type was given by Lexis in Ref. 15, pp. 161-163, where the computation is based on statistical data on the sex ratio at birth. There, Lexis gives a reference to Fechner who did the same thing earlier using psychological observations.

b. An interesting expository treatment of problems related to those considered in this paper has been given by Polyà[21]. It is notable that Polyà also considered there a game of chance leading to a hyperbolic partial differential equation - a matter which has to date received practically no attention in the literature. The author is indebted to Prof. Polyà for calling his attention to this reference and to the one in the preceding footnote.

Lewy[5] that the type of stochastic problem appropriate to elliptic differential equations is superficially quite different from the one which is naturally associated with parabolic equations. The reason is that, in the typical elliptic case, interest is centered on determining an unknown function throughout a bounded region in terms of known values given on a closed boundary. It is, therefore, to be expected that the boundary will somehow play a unique and peculiar role in the probabilistic formulation.

But actually, the peculiarity of the elliptic case is more apparent than real. The usual problem in stochastic processes may be described as follows: A process described by its transition probabilities runs on for m steps, or for a time t, where m or t are fixed in advance. Then the process is suddenly terminated. It is required to find the probabilities of the various terminal states. However, it is not considered to be out of the spirit of the game to place traps (e.g. absorbing barriers) along the way, which terminate the process automatically if it somehow falls into them. In the elliptic case, all possible terminal states consist of traps placed on the boundary of the region in question, and no time limit is set at all. Thus the elliptic problem is merely a special case of the usual problem with m or t infinite. Physically, it corresponds to the existence of a steady state.

These intuitively phrased considerations will be illustrated mathematically in Section 7 below.

The Gambler's Ruin 4

The starting point of the discussion will be a classical problem in games of chance, which, according to Uspensky,[a] was first solved by Huygens.

The problem is this: Two players G and G′ play a game consisting of a series of turns. The probability that G wins a turn is constant throughout the game. The stake in each turn is \$h. What is the probability that G ruins G′ before he himself is ruined?

The problem can be restated graphically. Let p be the probability that G wins a turn, and q the probability that G′ wins a turn; $p+q=1$. Let \$g be the wealth of G at the start of the game and let \$g′ be that of G′; it is assumed that g and g′ are integral multiples of h. Plot the points g and $b=g+g'$ on the x axis and divide up the interval $(0, b)$ into equal subintervals of length h.

A particle whose position represents the wealth of G at any moment in the game starts at g and performs a random walk in which the probability of a step to the right is always p and to the left is always q. Each step is of length h. What is the probability that the point reaches the right hand endpoint of the interval before it reaches the left hand endpoint?

Probabilities in infinite series of trials can be tricky, and so the existence of a solution requires attention first. Let x be the fortune of G at any point in the game, and let $v_m(x)$ denote the probability that he ruins G′ before getting ruined himself in, at most, m games, where m is for the moment fixed. This probability certainly exists, by any reasonable definition of probability. It can be broken down into two mutually exclusive cases: (1) G wins the first turn and ruins G′ within the next m - 1 turns; (2) G loses the first turn and ruins G′ within the next m - 1 turns. This leads to the difference equation

$$v_m(x) = p\, v_{m-1}(x+h) + q\, v_{m-1}(x-h) \quad 0<x<b ,$$

$$v_m(0) = 0 , \; v_m(b) = 1 . \tag{1}$$

This is a simple special case of the fundamental Chapman-Kolmogorov integral equation[b] of Markoff stochastic processes, which forms the starting point for many of the recent mathematical researches referred to in Section 2.

Now

$$v_{m+1}(x) - v_m(x) = p[v_m(x+h) - v_{m-1}(x+h)]$$
$$+ q[v_m(x-h) - v_{m-1}(x-h)] ,$$

so if we can show that $v_m(x) \geq v_{m-1}(x)$ for some m_0 and for all x, then it follows that this is true for all $m > m_0$. But $v_0(x) = 0,\; 0<x<b,$ and $v_0(x) = 1,\; x \geq b,$ and by (1),

$$v_1(x) = \begin{cases} p, & x = b - h \\ 0, & x < b - h \end{cases}$$

so the inductive chain is started with $m_0 = 1$. Therefore, $v_0, v_1, v_2, \ldots,$ is a monotonic sequence with unity as an upper bound. Thus, by a familiar result in analysis, $\lim v_m(x) = v(x)$ exists, $0 \leq x \leq b$. It is this limit which we call the probability of ruin.

It is a fact that if G does not ruin G′ in a game, then he will get ruined himself with probability 1;

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a. See Ref. 25, pp. 139, where existence of the solution is proved and formulas for the solution are derived for the case h = 1.

b. It can be written as

$$v_m(x) = \int_{-\infty}^{+\infty} v_{m-1}(\xi)\, d_\xi F(\xi|x) ,$$

where $F(\xi|x)$ denotes the probability that a single step starting at x in the random walk will make the particle reach the interval $\xi \leq x$.

that is, the probability that the game will go on forever is zero. We shall defer the proof until Section 11.

Taking the limit on both sides of (1) we obtain

$$v(x) = p\,v(x+h) + q\,v(x-h)\,, \qquad (2)$$
$$v(0) = 0,\ v(b) = 1\,.$$

The solution of this boundary value problem can be found by the usual methods of the theory of difference equations. It is:

$$v(x) = \frac{1-(p/q)^{-x/h}}{1-(p/q)^{-b/h}}\,,\quad q \neq p\,, \qquad (3a)$$

or

$$v(x) = x/b\,, \qquad q = p = \tfrac{1}{2}\,. \qquad (3b)$$

The answer to the original question is obtained by letting $x = g$ in these formulas. It is seen that if the wealth of G is large compared to that of G′, then G will have a good chance of ruining G′, even if each turn is disadvantageous to G.

Passage to the Differential Equation 5

It is natural to wonder what happens if the stake h in each turn is decreased while the initial wealth of each player is held constant. In the case $p = q = \frac{1}{2}$, formula (3b) seems to indicate that this would have no effect on the probability of ruin. If $p \neq q$, a glance at (3a) reveals that, if the stake is very small, the gambler to whom the turns are unfavorable almost surely is going to be ruined. To obtain interesting results in the case, the risks must be modified as the stakes grow smaller, so as to make them more nearly equal.

The situation is reflected in the difference equation (2) when passage to the corresponding differential equation is attempted. The equation can be written in terms of difference quotients, as follows:

$$\Delta^2 v(x) + \frac{p-q}{qh}\,\Delta v(x) = 0\,,\quad 0 < x < b \qquad (4)$$

where

$$\Delta v(x) = [\,v(x+h) - v(x)\,]/h \text{ and}$$

$$\Delta^2 v(x) = [\,\Delta v(x) - \Delta v(x-h)\,]/h$$

$$= [\,v(x+h) + v(x-h) - 2v(x)\,]/h^2\,.$$

Since formally as $h \to 0$,

$$\Delta v(x) \to \frac{dv}{dx} \text{ and } \Delta^2 v(x) \to \frac{d^2v}{dx^2}\,,$$

the difference equation goes formally into a differential equation if the coefficient of $\Delta v(x)$ approaches a limit.

An obvious way to arrange this is to stipulate that the odds shall be connected with the stakes in such a way that $p - q$ and h are of about the same magnitude. That is, let

$$p - q = h\alpha + o(h) \qquad (5)$$

where α is a constant and $o(h)$ denotes an infinitesimal of higher order than h. By using the relation $p + q = 1$, it is easy to show that (5) implies that

$$\frac{p}{q} = 1 + 2\alpha h + o(h)\,. \qquad (6)$$

Then the difference equation (4) goes over formally into

$$\frac{d^2u}{dx^2} + 2\alpha\frac{du}{dx} = 0\,, \qquad (7)$$

$$u(0) = 0\,,\ u(b) = 1\,.$$

The solution of (7) is easily found to be

$$u(x) = \frac{1-e^{-2\alpha x}}{1-e^{-2\alpha b}}\,.$$

Substituting (6) into (3a) and letting h approach zero, it is seen that

$$\lim_{h\to 0} v(x) = u(x)\,.$$

Therefore, not only does the difference equation formally go over into the differential equation, but the solution of one rigorously approaches that of the other.

In the case $p = q$, the second term in (7) is absent and the solution of the difference equation and differential equations are identical. This is to be anticipated from the fact that the solution of the difference equation was independent of h.

A More General Ordinary Differential Equation 6

The development in the preceding section proceeded in the classical way, i.e., from a probability problem to a functional equation, which was then solved analytically. But in this paper, the goal is to investigate the possibility of reversing the argument. It is clear that the probability problem gives an exact solution of the boundary value problem (2) or (4) and an approximate solution of the boundary value problem (7) if h is small and p and q are nearly equal. Moreover, if the game were played over and over many times, and the relative frequency of times in which G′ was ruined rather than G were recorded, this relative frequency would be an estimate of the solution at the point $x = g$.

Thus, the game clearly provides a method for approximate numerical quadrature of (7). The remainder of the paper is devoted to study of the method.

Consider, now, the boundary value problem given by

$$\mathcal{L}(U) \equiv \beta\frac{d^2U}{dx^2} + 2\alpha\frac{dU}{dx} = 0\,, \qquad (8)$$

$$U(a) = A\,,\quad U(b) = B\,,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are uniformly bounded functions of x with $\beta(x) > d > 0$ for $a < x < b$.[a] Let a lattice or mesh of equally spaced points now be superimposed on the x-axis, dividing it into subintervals of length h. (The number h is called the mesh constant of the lattice.) A problem in difference equations corresponding to (8) is

$$L(V) \equiv \beta \Delta^2 V + 2\alpha \Delta V = 0 , \quad a < x < b , \tag{9}$$

$$V(x) = \begin{cases} A, & x \leq a \\ B, & x \geq b \end{cases} .$$

Rearranging the terms and solving for V(x), we obtain

$$V(x) = p(x)\, V(x+h) + q(x)\, V(x-h) , \quad a < x < b , \tag{10}$$

$$V(x) = \begin{cases} A, & x \leq a \\ B, & x \geq b \end{cases} ,$$

where

$$p(x) = \frac{\beta(x) + 2h\,\alpha(x)}{D(x)}$$

$$q(x) = \frac{\beta(x)}{D(x)}$$

$$D(x) = 2\,\beta(x) + 2h\,\alpha(x) .$$

(If $\alpha = \alpha(x)$ is sometimes negative, we assume that $2h\,|\alpha| < d$, $a < x < b$.)

It will be noted that $p(x) + q(x) = 1$ and that (10) looks very much like (2). This suggests that a probability model analogous to the problem of the gambler's ruin exists for (10), and that p and q can again be interpreted as the transition probabilities in a random walk.

In fact, consider the following situation: A particle starts at a point x_0 lying on the lattice and inside the interval $[a, b]$, and performs a random walk on the lattice. The conditions of the walk are that if the particle is momentarily at the point x, the probability of a step of length h to the right is p(x) and the probability of a step of length h in the other direction is q(x), where p and q are given by the above formulas. If the particle reaches the interval $x \leq a$ or the interval $x \geq b$, the walk is stopped. In the former case, a score A is tallied; in the latter case, a score B is tallied. What is the mean value of the score?

Let $v_m(x, b)$ be the probability of reaching the interval $x \geq b$ before arriving at $x \leq a$ if the walk is allowed to continue for only m steps, and let $v_m(x, a)$ be the probability of reaching $x \leq a$ before reaching $x \geq b$ in, at most m steps. Let $V_m(x)$ be the mean value if, at the end of the walk, B is tallied for $x \geq b$, and A is tallied for $x \leq a$, and 0 is tallied if the particle is still in the interval $[a, b]$ after m steps. Then

$$V_m(x) = Bv_m(x, b) + Av_m(x, a) .$$

The function $v_m(x, b)$ satisfies a Chapman-Kolmogorov equation:

$$v_m(x, b) = p(x)\, v_{m-1}(x+h, b) + q(x)\, v_{m-1}(x-h, b), \quad a < x < b , \tag{11}$$

$$v_m(x, b) = \begin{cases} 0, & x \leq a \\ 1, & x \geq b \end{cases}$$

and a similar one is satisfied by $v_m(x, a)$. Since $V_m(x)$ is a linear combination of functions satisfying the linear equation (11), $V_m(x)$ itself satisfies a similar Chapman-Kolmogorov equation:

$$V_m(x) = p(x)\, V_{m-1}(x+h) + q(x)\, V_{m-1}(x-h), \quad a < x < b, \tag{12}$$

$$V_m(x) = \begin{cases} A, & x \leq a \\ B, & x \geq b \end{cases} .$$

Just as in Section 4, it can now be shown that

$$\lim_{m \to \infty} v_m(x, b) = v(x, b)$$

exists, and $\displaystyle\lim_{m \to \infty} v_m(x, a) = v(x, a)$

exists, and, therefore,

$$\lim_{m \to \infty} V_m(x) = V(x) = Bv(x, b) + Av(x, a) \quad \text{exists.}$$

Therefore, taking the limit in (12), it is seen that V(x) satisfies (10) and (9).

The mean value V(x) considered in the preceding paragraph is the mean value of scores A, B and 0, tallied respectively if the particle reaches $x \leq a$ before $x \geq b$, or $x \geq b$ before $x \leq a$, or never reaches either $x \leq a$ or $x \geq b$. But it will be shown in Section 11 that the last case occurs with zero probability, so the only cases which enter into the mean value are the first two.

Thus, it is seen that the mean value of the tally in the random walk problem furnishes a solution to the boundary value problem (9) at the point x_0.

Several comments are now in order.

1. As in the limiting case of the gambler's ruin, it will be noticed that $p(x) - q(x)$ is of the same order as h; in fact,

$$p(x) - q(x) = h\,\frac{\alpha(x)}{\beta(x)} + o(h).$$

It is to be noted that the difference equation (9) when placed in the form (10), seems automatically to select the probabilities of the appropriate equivalent

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a. It is assumed whenever a differential equation is written down in this paper that the coefficients possess sufficient regularity for the solution to exist.

game. This is true also for the cases of higher dimensionality.

2. It is of interest to observe that, inasmuch as $V(x)$ is the weighted mean of $V(x + h)$ and $V(x - h)$, it must lie between these two values. Therefore, it can have no relative maxima and minima inside the interval $[a, b]$, and its value must always lie in the interval $[A, B]$ or $[B, A]$ defined by the boundary values. This means that if $A = B = 0$, then $V(x) \equiv 0$. Thus there cannot be two solutions $V(x)$ and $V^*(x)$, $V(x) \not\equiv V^*(x)$, both satisfying (10) with the same boundary values, because their difference would satisfy (10) with zero boundary values, and would then be identically zero.

3. In carrying out the successive approximations defined by (12), it is actually possible to start with any initial approximation $V_0(x)$ (see Ref. 23, pp. 184-185). This can be used as the basis for a numerical procedure known as the Liebmann Method, which is particularly useful in a higher number of dimensions. The Liebmann Method, of course, does not involve probability consideration.

4. There is another interpretation of (10) which can also be used as the basis of a non-probabilistic numerical technique. It is as follows: A mass of substance concentrated originally at x begins to diffuse along the x-axis. During the first second, $p(x)$ of the material goes to $x + h$, and $q(x)$ goes to $x - h$. During the second second, $p(x + h)$ of the material at $x + h$ goes to $x + 2h$, and $q(x + h)$ goes to $x + h - h = x$. Also $p(x - h)$ of the material at $x - h$ goes to x and $q(x - h)$ goes to $x - 2h$. This continues until all the material has left the interval $a < x < b$. The proportion reaching $x \geq b$ is noted; let this be P_b. The proportion reaching $x \leq a$ is also noted; let this be P_a. Then $V(x) = A\,P_a + B\,P_b$. When reduced to computing practice this method has been called the "census method"[19].

A Related Parabolic Differential Equation 7

The case in which the random walk is restricted to, at most, m steps will now be considered again. A time variable t is now introduced by the expedient of letting $t = m\lambda$, where λ is interpreted as the time interval for a step of the walk. Let $V(x,t) \equiv V_m(x)$. It is instructive to write (12) in the equivalent form

$$\frac{V(x,t+\lambda) - V(x,t)}{\lambda} = \frac{\beta}{D}\left[\frac{V(x+h,t) + V(x-h,t) - 2V(x,t)}{\lambda}\right] \tag{13}$$

$$+ \frac{2h\alpha}{D}\left[V(x+h,t) - V(x,t)\right],\ t > 0,\ a < x < b,$$

$$V(x,t) = \begin{cases} A, & x \leq a,\ t \geq 0 \\ B, & x \geq b,\ t \geq 0 \\ 0, & a < x < b,\ t = 0. \end{cases}$$

If $\lambda = h^2$, and then $\lambda \to 0$, this equation goes formally over into the parabolic differential equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} + \frac{\alpha(x)}{\beta(x)}\frac{\partial V}{\partial x}. \tag{14}$$

It can, in fact, be proved under suitable restrictions that the solution of (14) satisfying the boundary conditions stated above does exist, and that the limiting solution of (12) or (13) is the solution of this equation (see Ref. 13, pp. 24-31 and Ref. 9). The equation is closely related to the Fokker-Planck equation of mathematical physics. In the diffusion interpretation of the random walk with $A = B = 1$, the equation yields an expression for the rate $\frac{\partial V}{\partial t}$ at which the material disappears into the boundary of the infinite space-time strip $a < x < b$, $t \geq 0$, as time goes on.

If, now, in (13), we let $t \to \infty$ first, with a fixed $\lambda = h^2$, then the left side approaches zero, and, in the limit, the equation is identical with (9). The formal limit as $\lambda \to 0$ is now the differential equation (8). It is again possible to prove that the form has substance: as $\lambda \to 0$ the solution of (9) does approach the solution of (8). This will be discussed more fully in the next section.

In later sections it will be seen that the natural generalization of the problem considered in Section 6 to several space variables x, y, z, . . . , consists in the replacement of the expression on the left side of (8) (or the right side of (14)) by an elliptic differential operator. The developments of the present section, although purely formal, show why the problems considered in this paper may be considered as limiting cases of the possibly more familiar probability problems associated with parabolic operators. The relationship was previously discussed in general terms in Section 3.

Generalization of the Stochastic Process and Passage to the Differential Equation 8

It can be proved in various ways that, as h approaches zero, the solution of the difference equation (9) approaches that of the differential equation (8). (See Ref. 10, pp. 160-166.) A completely general proof will not be given in this paper, although Section 13 below contains the essence of such a proof.

In considering the asymptotic characteristics of the random walk of Section 6, it is natural to inquire whether it provides the only probabilistic approach to the solution of (8). The question was answered in the negative by Petrowsky[20], who established an asymptotic connection between a broad class of stochastic processes and (8). His result is, in a sense, fundamental to this paper. Although limitations of space preclude giving the details of his proof here (it is rather complicated, but is readily accessible in Ref. 13), it, nevertheless, seems to be desirable, at least, to state his theorem and show that it incidentally establishes that the solution of (9) approaches that of (8).

A parameter λ is again introduced, which may be considered to represent the time interval between observations on the state of the process. The asymptotic character of the process will now be made to depend entirely on λ, and the lattice previously used to set up the difference equation in Sections 4 and 6 will not play a role here at all. The states will again be represented by the successive abscissas of a moving particle on the x-axis, and thus the process may be considered as a general form of random walk. The length and direction of the kth step taken by the particle will be denoted by a random variable X_k.

It will be assumed that the distribution of X_k depends only on λ and on the position x of the initial point from which the step is taken, but not on k nor in any other way on the past history of the process. This is the characteristic property of a Markoff process.

The distribution of X_k is called the transition distribution of the stochastic process. The distribution function of the transition distribution will be denoted by $F_\lambda(\xi|x)$; this is the probability that X_k is less than or equal to $\xi - x$ if the kth step started at x. We assume (merely to simplify statements) that $F_\lambda(\xi|x)$ is defined for all x, $a < x < b$.

The first and second moments about the point x and third absolute moment of the transition distribution will be denoted respectively by $M_\lambda(x)$, $D_\lambda(x)$, $T_\lambda(x)$; they are given by the formulas

$$M_\lambda(x) = \int_{-\infty}^{+\infty} (\xi - x)\, d\xi\, F_\lambda(\xi|x) \ ,$$

$$D_\lambda(x) = \int_{-\infty}^{+\infty} (\xi - x)^2\, d\xi\, F_\lambda(\xi|x) \ ,$$

$$T_\lambda(x) = \int_{-\infty}^{+\infty} |\xi - x|^3\, d\xi\, F_\lambda(\xi|x) \ .$$

Now let the random walk start at a point x on the interval $a < x < b$ and proceed until it reaches the interval $x \geqslant b$ or the interval $x \leqslant a$. If it reaches the interval $x \leqslant a$ first, a quantity A is tallied. If it reaches $x \geqslant b$ first, a quantity B is tallied. Let V(x) be the mean value of the tally, assumed for the moment to exist. It is to be noted that since the transition distribution depends on λ, so does V(x).

The theorem is then as follows:

Theorem 1:[a] Let $M_\lambda(x)$, $D_\lambda(x)$, $T_\lambda(x)$ exist for $a < x < 0$ and let the distribution of X_k be such that

$$\begin{aligned} M_\lambda(x) &= \lambda\, k(x)\, \alpha(x) + o(\lambda) \\ D_\lambda(x) &= \lambda\, k(x)\, \beta(x) + o(\lambda) \qquad\qquad (a) \\ T_\lambda(x) &= o(\lambda), \end{aligned}$$

uniformly in x, where $\alpha(x)$, $\beta(x)$ and k(x) are continuous functions and $\beta(x) > 0$, $k(x) > 0$, $a \leqslant x \leqslant b$. Furthermore, let there exist positive numbers ϵ_λ and η_λ independent of x such that

$$1 - F_\lambda(x + \epsilon_\lambda | x) = \text{Prob}\,(X_k > \epsilon_\lambda | X_k = x) \geqslant \eta_\lambda . \quad (b)$$

[b]

Then as $\lambda \to 0$, the mean value V(x) approaches the solution U = U(x) of the boundary value problem

$$\beta(x) \frac{d^2U}{dx^2} + 2\alpha(x) \frac{dU}{dx} = 0,$$

$$U(a) = A, \ U(b) = b,$$

uniformly in the interval $a \leqslant x \leqslant b$.

It is easily seen that the random walk considered in Section 6 satisfies the condition of the theorem, provided that $\lambda = h^2$ and $\alpha(x)$ and $\beta(x)$ have the smoothness properties required by the theorem. For

$$M_\lambda(x) = h\, p(x) - h\, q(x) = \frac{h^2\alpha}{\beta + h\alpha} = h^2 \frac{1}{\beta} \cdot \alpha + o(h^2),$$

$$D_\lambda(x) = h^2\, p(x) + h^2\, q(x) = h^2 = h^2 \cdot \frac{1}{\beta} \cdot \beta$$

$$T_\lambda(x) = h^3\, p(x) + h^3\, q(x) = h^3 = o(h^2).$$

Setting $\lambda = h^2$ and letting $k(x) = 1/\beta(x)$, we see that the condition (a) in the statement of the theorem is satisfied. Also

$$\text{Prob}\,(X_k > \frac{h}{2} | x) = p(x),$$

which is bounded from zero for all x because the numerator of p(x) is supposed to have that property. Therefore, condition (b) is satisfied with

$$\epsilon_\lambda = h/2 = \sqrt{\lambda}/2 \text{ and } \eta_\lambda = p(x) = (\beta + 2\sqrt{\lambda}\alpha)/(2\beta + 2\sqrt{\lambda}\alpha).$$

It will be noted that the starting point of the walk in Section 6 was not restricted to any particular denumerable sequence of lattice points, but rather was

---

a. Petrowsky[20] proves the theorem with the condition on the third moment given here replaced by a Lindeberg condition, which is weaker than the present one. He considers only the case in which $k(x) = 1$, $A = 0$, $B = 1$; but the extension to the present case is trivial.

b. By Prob. (M|N) is meant the conditional probability of the event M, given that N has happened. Condition (b) means that the probability is at least η_λ in any step that starts at x, in which the abscissa of the moving particle is increased by more than ϵ_λ.

permitted to be any point of the interval. This implies that for each h the transition distribution function of the process is defined for all values of x in [a, b].

Let V(x, t) be the expected tally if the random walk is limited to, at most, $t = m\lambda$ steps, and if A is scored when the region $x \leq a$ is reached, B is scored when $x \geq b$ is reached, and 0 is scored if the particle is still in the interval $a < x < b$ at time t. Then, exactly as in Section 6, a Chapman-Kolmogorov equation is obtained:

$$V(x,t+\lambda) = \int_{-\infty}^{+\infty} V(\xi,t)\, d_\xi F_\lambda(\xi|x), \qquad (15)$$

$$t \geq 0,\ a < x < b,$$

$$V(x,t) = \begin{cases} A, & t \geq 0,\ x \leq a \\ B, & t \geq 0,\ x \leq b \\ 0, & t = 0,\ a < x < b. \end{cases}$$

The existence of the limit of V(x)[a] as $t \to \infty$ follows by induction in the same way as before. It will be shown in Section 11 that the probability of remaining in $a < x < b$ is zero. The limit function satisfies an equation similar to (15), and the proof of Theorem 1 consists in showing that this equation is asymptotically equivalent to the differential equation.

Equation (15) may be written as follows:

$$\frac{V(x,t+\lambda) - V(x,t)}{\lambda} = \frac{1}{\lambda} \int_{-\infty}^{+\infty} [V(\xi,t) - V(x,t)]\, d_\xi F_\lambda(\xi|x).$$

In this form it is a direct analogue of (13). Under suitable conditions (see Ref. 13, pp. 24-31) as $\lambda \to 0$ the integral equation will go formally over into the parabolic differential equation (14) and the solution of the one will equal that of the other.

General Dirichlet Problems for Differential and Difference Equations 9

The results of the preceding sections are of some theoretical interest, but, probably, have little significance for practical numerical analysis. Fortunately, however, they all generalize immediately to spaces of two or more dimensions, where their importance for computational techniques is much greater because of the well-known difficulties of attacking partial differential equations in many variables by numerical methods.

For simplicity, the exposition here will be restricted principally to two dimensions. The arguments and results carry over into spaces of higher dimensionality without essential change.[a]

The central problem will be the following one, which we shall call Problem **D**:

Let C denote a Jordan curve[b] of the finite plane; let R denote the region[c] interior to C;

let $\bar{R} = R + C$; let $\phi(x,y)$

be a given function piecewise continuous on C. Required, to find for all points in R that solution U(x, y) of the elliptic differential equation

$$\mathscr{L}(U) \equiv \beta_{11}\frac{\partial^2 U}{\partial x^2} + 2\beta_{12}\frac{\partial^2 U}{\partial x \partial y} + \beta_{22}\frac{\partial^2 U}{\partial y^2} + 2\alpha_1\frac{\partial U}{\partial x} + 2\alpha_2\frac{\partial U}{\partial y} = 0 \qquad (16)$$

for which $U(x,y) \equiv \phi(x,y)$, (x, y) on C, where β_{11}, $\beta_{12}, \beta_{22}, \alpha_1, \alpha_2$, are functions of x and y continuous and with continuous first and second derivatives in a region containing R, with

$$\beta_{11} > 0,\ \beta_{22} > 0,\ \beta_{11}\beta_{22} - \beta_{12}^2 > 0 \text{ in } \bar{R}.$$

This is the Dirichlet problem, or first boundary value problem of potential theory, for a homogeneous elliptic differential equation of a rather general type.

A rectangular lattice of points of the (x, y) plane is now selected, consisting of the vertices of squares of side h formed by the system of lines

$$x = x_0 + jh,\ y = y_0 + jh,\ j = 0, \pm 1, \pm 2, \ldots$$

Here x_0 and y_0 are chosen conveniently, and h is the mesh constant.

If h is sufficiently small, (and we henceforth assume this to be the case) then the lattice points in R will form a single, uniquely determined lattice region R_h, with the property that any point of R_h can be connected to a fixed preassigned point of R_h by a chain of lattice points lying in R_h. By a chain of lattice points is meant a sequence of lattice points in which each point (after the first) follows one of its four neighbors (i.e., the four points at distance h).[d]

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a. It should be noted that both V(x) and V(x, t) for fixed t depend on λ because the transition distribution does.

a. It is true that there are important differences in the theory in two dimensions and in higher dimensions when infinite regions are considered, as is pointed out in (21). But here we shall be concerned only with bounded regions.

b. That is, a one-to-one continuous transform of a circumference.

c. By region is meant an open connected set.

d. It is not assumed that the line segments joining successive points in a chain must lie entirely in R, although this will be true if h is small enough.

Certain neighbors of some of the points in R_h will not be included in R_h. Furthermore, the point $(x + h, y + h)$ will not be in R_h for all (x, y) in R_h. Let C_h be the set of missing neighbors plus the missing points $(x + h, y + h)$, belonging to all points in R_h. Then the set $\bar{R}_h = R_h + C_h$ has the following properties:

1. Every lattice point in R is included in R_h.
2. All points of C_h lie on or outside of C.
3. Each point (x, y) of R_h is an interior point, in the sense that all four of its neighbors and the point $(x + h, y + h)$ are present in $R_h + C_h$.[a]
4. Any point in R_h can be connected to any point in $\bar{R}_h$ by a chain of neighbors, which lie in R except possibly for the last point in the chain.

It is of interest to note that, given any point (x, y), if h is sufficiently small, a uniquely determined lattice region R_h containing (x, y) with properties (1), (2), (3), (4) exists for all smaller values of h.[b] If the point (x, y) is restricted to lie on an arbitrary but fixed closed point set interior to R, then there is a minimum value of $h > 0$, such that this situation holds for all $h < h_0$ uniformly for every point on the closed point set.

Differences in the x and y directions will be symbolized as follows:

$$\Delta_x V = \frac{V(x + h, y) - V(x, y)}{h},$$

$$\Delta_{xx} V = \frac{V(x + h, y) + V(x - h, y) - 2V(x, y)}{h^2},$$

$$\Delta_{xy} V = \frac{V(x + h, y + h) - V(x + h, y) - V(x, y + h) + V(x, y)}{h^2},$$

$$\Delta_y V = \frac{V(x, y + h) - V(x, y)}{h},$$

$$\Delta_{yy} V = \frac{V(x, y + h) + V(x, y - h) - 2V(x, y)}{h^2} .$$

The argument (x, y) will frequently be denoted by P or Q. The neighbors of P in counter-clockwise order, starting with $(x + h, y)$ will be denoted by subscripts 1, 2, 3, 4 and the point $(x + h, y + h)$ by the subscript 5.

a. Some of the points of C_h may be interior points in this sense also.

b. A proof of the existence of R_h with properties (1) - (4) for some h sufficiently small and for all values of h smaller, can be given along the lines indicated in Ref. 29, pp. 7-9, where it is shown that the region R can be approximated arbitrarily closely by a simply connected region formed out of squares of side h.

Finally, we let

$$L(V) = \beta_{11} \Delta_{xx} V + 2\beta_{12} \Delta_{xy} V + \beta_{22} \Delta_{yy} V + 2\alpha_1 \Delta_2 V + 2\alpha_2 \Delta_y V ,$$

where the coefficients of the differences have the same properties as the coefficients of (16).

In the difference equation formulation, Problem D now becomes the following, which we call Problem D_h:

Let $\phi(x, y)$ be given on C_h. Required to solve the difference equation

$$L(V) = 0 \tag{17}$$

for (x, y) on R_h, subject to the boundary conditions that

$$V(x, y) = \phi(x, y) \text{ on } C_h .$$

Rearranging the terms of (17) and solving for $V(P)$, we obtain

$$V(P) = p_1(P)\, V(P_1) + p_2(P)\, V(P_2) + p_3(P)\, V(P_3) + p_4(P)\, V(P_4) + p_5(P)\, V(P_5) , \tag{18}$$

where

$$p_1(P) = \frac{\beta_{11} - 2\beta_{12} + 2h\alpha_1}{D}$$

$$p_2(P) = \frac{\beta_{22} - 2\beta_{12} + 2h\alpha_2}{D}$$

$$p_3(P) = \frac{\beta_{11}}{D}$$

$$p_4(P) = \frac{\beta_{22}}{D}$$

$$p_5(P) = \frac{2\beta_{12}}{D}$$

and

$$D = D(P) = 2\beta_{11} + 2\beta_{22} - 2\beta_{12} + 2h(\alpha_1 + \alpha_2) .$$

As in the one dimensional case $\sum_1^5 p_j(P) = 1$, but here $p_1(P)$ or $p_2(P)$ may be negative if $2\beta_{12} > \beta_{11}$ or $2\beta_{12} > \beta_{22}$. We, therefore, assume in the sequel that

$$2\beta_{12} < \beta_{11} \text{ and } 2\beta_{12} < \beta_{22} \text{ for P in } R_h.\text{[a]}$$

It is also supposed that h is so small that if α_1 and α_2 are sometimes negative, the quantities $p_j(P)$ are nevertheless all non-negative in R_h.

Consider now the following problem: Let the points of C_h be denoted by Q_j, $j = 1, 2, \ldots$. A particle starts at the point P_0 on R_h and performs a random walk (or random flight) on R_h. The conditions of the

a. It is to be noticed that in any case, $\beta_{12}^2 < \beta_{11}\beta_{22}$ implies that $2\beta_{12} < \beta_{11} + \beta_{22}$ by the inequality for the geometric and arithmetic means.

walk are that if the particle is momentarily at the arbitrary point P, the probabilities of stepping to

P_1, P_2, P_3, P_4, P_5 are $p_1(P), p_2(P), p_3(P), p_4(P)$,

and $p_5(P)$ respectively. When the particle reaches a point Q_j the walk is terminated and $\phi(Q_j)$ is tallied. What is the mean value of the score?

Let ϕ_0 be the absolute value of a lower bound for $\phi(Q_j)$ for Q_j on C_h. The quantity $\phi(Q_j) + \phi_0$ is never negative. The required mean value, if it exists, is the difference of the corresponding mean values if the tally at Q_j is $\phi(Q_j) + \phi_0$, and if the tally at Q_j is ϕ_0, provided that these mean values can be shown to exist. Thus it suffices to establish the existence of the mean value for a non-negative set of boundary values, say $\psi(Q_j)$, which will now be done.

First, let the walk be limited to m steps at most. Let $V_m(P)$ be the mean value if $\psi(Q_j)$ is tallied if the walk reaches Q_j, $j = 1, 2, \ldots$, and zero if the particle is still on R_h after m steps.

The function $V_m(P)$ obviously satisfies for each P on R_h a Chapman-Kolmogorov equation:

$$\begin{aligned} V_m(P) &= p_1(P)\,V_{m-1}(P_1) + p_2(P)\,V_{m-1}(P_2) \\ &\quad + p_3(P)\,V_{m-1}(P_3) + p_4(P)\,V_{m-1}(P_4) \\ &\quad + p_5(P)\,V_{m-1}(P_5)\,, \qquad P \text{ on } R_h\,, \end{aligned} \tag{19}$$

$$V_m(Q_j) = \psi(Q_j)\,, \quad j = 1, 2, \ldots \; .$$

Then

$$V_{m+1}(P) - V_m(P) = \sum_{j=1}^{5} p_j(P)[V_m(P_j) - V_{m-1}(P_j)]\;,$$

so, if for any m, $V_{m+1} \geq V_m$ for all P on R_h, the inequality will be true for all greater values of m. But $V_0(P) = 0$ for P on R_h, and $V_0(Q_j) = \psi(Q_j)$, $j = 1, 2, \ldots$. Since the coefficients p_1, p_2, p_3, p_4, p_5, are non-negative, it will follow that $V_1(P) \geq V_0(P)$ for all P on R_h. Also $0 \leq V_m \leq \overline{\text{bound}}\ \psi(Q_j)$. Therefore, the sequence $\{V_m\}$ is a monotonic bounded sequence and must have a limit for every P on R_h. Taking the limit in (19), it is seen that the limiting function V(P) satisfies (17) and (18).

It will be shown in Section 11 that if $m = \infty$, the probability of tallying a zero is, itself, zero. Therefore, the mean value of the tally in the random walk problem with the number of steps unlimited furnishes a solution to Problem D_h.[a]

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a. McCrea and Whipple[17] have derived explicit expressions for V(P) in the case in which R_h is rectangular, $\phi(Q_1) = 1$, $\phi(Q_j) = 0$, $j = 2, \ldots$, $L(V) \equiv \Delta V \equiv \Delta_{xx}V + \Delta_{yy}V$. They also give explicit expressions for the mean number of visits to a given point P' on R_h. The three-dimensional case and the case of an unbounded R_h are considered.

All of the comments made in Section 6 are again applicable, with slight changes in the wording to take care of dimensionality. In particular, V(P) can have no relative maximum nor minimum on R_h and its value must always lie in the closed interval bounded by the greatest and least value of $\phi(Q_j)$. If $\phi(Q_j) \equiv 0$, then $V(P) \equiv 0$; the solution of Problem D_h is accordingly unique.

Generalization of the Stochastic Process and Passage to the Partial Differential Equation 10

As in the one-dimensional case, the function V(P), under certain conditions, approaches the solution of Problem D as $h \to 0$. A standard proof is the one given by Courant, Friedrich and Lewy.[5] In this reference, the details are given only for the case in which $L(V) = \Delta V = \Delta_{xx}V + \Delta_{yy}V$, but the proof generalizes to any second-order elliptic difference equation. In more than two dimensions it establishes, only, that the boundary value $\phi(x, y)$ are approximated in the mean.

Feller (see Ref. 23, pp. 187-194) reproduces a proof given in 1941 by Petrowsky for the case $L(V) = \Delta V$. The proof proceeds[5] by showing, first, that for a sequence of values of h approaching zero, the sequence of solutions of $\Delta V = 0$ is equicontinuous in $\bar{R}_h$; but the Petrowsky proof then goes on to establish, in a way which is valid for any dimensionality, that the limit function actually assumes the given boundary values as P approaches a point Q on C.

Both of these proofs have the mathematical advantage that they furnish proofs of the existence of a solution to Problem D. But if the point of view is adopted that the existence and uniqueness of the solution to Problem D has already been proved by one method or another, and that the problem is to decide when a random walk will approximate the solution, then a formulation of the asymptotic situation can be given which is the direct generalization of Theorem 1 to two or more dimensions. This was done in 1933 by Petrowsky.[20] His result is a generalization of an investigation made a year or two earlier by Lüneberg.[16]

The essence of the Petrowsky result in two dimensions is this: The states of a stochastic process are represented by the coordinates of a particle moving in the (x, y) plane, whose position is observed at time interval λ. The transition distribution describing the passage from one state to another is defined by a distribution function $F_\lambda(\xi, \eta | x, y)$ which given the conditional probability that the x and y coordinates of the particle do not exceed ξ and η respectively after a step which started at the point (x, y). This distribution depends only on (x, y) and λ, and not on past history, so the process is Markoffian.

As in Theorem 1, it is assumed that the first and second moments of F about x and y are of the order of λ. A further condition, such as the one on T_λ in Theorem 1, is added to insure that the tails of the transition distribution approach zero with λ at a suitable speed. Also, a restriction analagous to condition (b) in Theorem 1 is imposed on F_λ so as to insure that the probability of the particle remaining in R indefinitely is zero (see Theorem 2 in the next section).

It is assumed that $\phi(x, y)$ is uniformly bounded and continuous everywhere in the complement of R, or is at least piecewise continuous there with simple discontinuities along non-intersecting arcs connecting C to infinity.

Suppose, now, that the random walk starts at a point P in R and proceeds until it first reaches a point Q, not in R, whereupon $\phi(Q)$ is tallied. Let V(P) be the mean value of the tally; this mean value depends on λ because the transition distribution does. The theorem of Petrowsky asserts that under the conditions here outlined, V(P) approaches the solution of Problem D as $\lambda \to 0$.

It turns out, as in the one-dimensional case, that the special random walk considered in Section 9 satisfies the hypothesis of the theorem.

Thus, there are an infinity of random walk problems equivalent asymptotically to the one in Section 9. In fact, given any Markoffian random walk in a bounded region whose boundaries are treated as absorbing barriers, if the first two moments of the transition distribution are both of the order of the time interval required for a single transition, and if the "value" of each walk corresponds to where the walk ends on the boundary, then the mean value of the walk will ordinarily approximate the solution of a Dirichlet problem for the region.

The theorem is proved in two dimensions in Ref. 20 with the added hypothesis that C consists of a finite number of arcs, each with continuous curvature. In the two-dimensional case, this restriction on C can be removed. However, in the case of three and more dimensions, it is well-known that Problem D does not have a solution unless the boundary C is suitably restricted beyond the mere requirement that it be a one-to-one continuous map of the surface of a sphere. A sufficient condition (see Ref. 12, p. 329) is that each point Q of C is the vertex of any right circular cone, which has no points in the part of R lying in a suitably small spherical neighborhood of Q. Inspection of Petrowsky's proof reveals that it can be modified so as to require no more than this condition on C in three and more dimensions.

The Analogue of Green's Function 11

The notation used in the preceding sections for the one and the two dimensional cases will now be consolidated. In one dimension, R will denote the interval $a < x < b$, R_h will denote the points of the lattice described in Section 6 which lie in the interval

$$a < x < b,$$

C_h will consist of the two points of the lattice not in $a < x < b$ but nearest to the interval (one in $x \leq a$ and one in $x \geq b$), P will denote any point of $a < x < b$ and Q any point of the complement of this interval with respect to the x-axis, C will consist of the points $x = a$ and $x = b$; and the boundary value function $\phi = \phi(x)$ will have only two values: $\phi(x) = A$, $x \leq a$, $\phi(x) = B, x \geq b$. Problem D will mean the one given by (8) and Problem D_h the one given by (9). Also it is to be understood in the one dimensional case that $p_1(P) = p(x)$, $p_2(P) = q(x)$, $p_3 = p_4 = p_5 = 0$. On the other hand, in the two-dimensional case a and b will be defined to mean respectively the greatest lower bound of x for (x, y) in R and the least upper bound of x for (x, y) in R. The random variable X_k will denote the x component of the kth step of the walk.

Let the steps in a random walk of any of the types considered in the preceding sections be numbered serially, and let the random variable n denote the serial number of the step which ends the walk. The mean value of n will be denoted by $E(n|P_0)$; it depends on λ or h as well as on the starting point P_0 of the walk.

Let k be the integral part of the number $1 + (b - a)/\epsilon$ where ϵ is identified in the statement of the following theorem.

Theorem 2: If for fixed λ or h, positive numbers ϵ and η exist which are independent of P, such that

$$\text{Prob}\ (X_k > \epsilon \mid P) > \eta \tag{20}$$

for all P in R, then

(a) $\text{Prob}\ (n > km) < (1 - \eta^k)^m$, $m = 0, 1, \ldots,$

(b) $\text{Prob}\ (n = \infty) = 0$,

(c) $E(n|P_0) < k/\eta^k$, and is therefore uniformly bounded for all P_0 in R.

To prove the theorem, first consider

$$\text{Prob}\ (n \leq k) = 1 - \text{Prob}\ (n > k).$$

This is the probability that the walk leaves R in, at most, k steps. Let δ be the distance from the starting point P of the walk to the nearest boundary point as measured in the direction of increasing x along a line drawn through P parallel to the x-axis. Then since $\delta < b - a$,

$$\text{Prob}\ (n \leq k) \geq \eta^{1 + (\delta/\epsilon)} > \eta^k .$$

Therefore

$$\text{Prob}\ (n > k) < 1 - \eta^k .$$

Now $\text{Prob}(n > mk)$, $m > 1$, is equal to the probability that the walk continues inside R for k steps, multiplied by the probability that it does so for k more, etc., and so on for m factors. That is

$$\text{Prob } (n > mk) < (1 - \eta^k)^m,$$

as was to be proved for (a).

Since Prob $(n > j)$ decreases monotonically with j, (b) follows at once.[a] Finally,

$$E(n|P_0) = \sum_{j=0}^{\infty} j \text{ Prob } (n = j) = \sum_{j=0}^{\infty} \text{Prob } (n > j)$$

$$\leqslant \sum_{j=0}^{\infty} k \text{ Prob } (n > kj) < k \sum_{j=0}^{\infty} (1 - \eta^k)^j$$

$$= \frac{k}{\eta^k}.$$

Unfortunately, the bound for $E(n|P_0)$ given by the theorem is not in general a good one.[b]

In the case of a random walk on a lattice R_h, condition (20) need be valid only for all P on R_h for the theorem to be true. In two dimensions, condition (20) could, of course, be stated in terms of the y component of a step in the walk instead of the x component, provided that the meanings of a and b were properly adjusted.

The random walk on R_h considered in Sections 6 and 9 will now be further studied. Let P and P′ be two points of R_h, and let $G(P, P')$ be the mean value of the number of times the particle visits P′ in the course of the random walk if it started at P. Let

$$\delta(P, P') = 0 \text{ if } P \neq P' \text{ and } \delta(P, P) = 1.$$

Theorem 3: The function $G(P, P')$ satisfies the difference equation in P,

$$L(G) = \frac{-D(P')}{h^2} \cdot \delta(P, P')$$

with $G(Q, P') = 0$ for Q on C_h.

Here $D(P')$ is the function D defined in connection with (10) in one dimension and (18) in two. The proof consists of showing that $G(P, P')$ exists and satisfies the equation

$$G(P, P') - \delta(P, P') = \sum_{1}^{5} p_j(P) \, G(P_j, P'),$$

which is the same as the equation in the theorem.

For the proof, let $P_i(P, P')$ denote the probability that the random walk ends on P′ in exactly i trials, i = 0, 1, 2, ... We note that $P_i(Q, P') = 0$ for Q on C_h. Then

$$G(P, P') = \sum_{i=0}^{\infty} P_i(P, P'), \tag{21}$$

provided that this series can be shown to converge. Now $P_{mk}(P, P')$ is certainly not greater than the probability that the walk is still inside the interval after mk steps. That is,

$$P_{mk}(P, P') \leqslant \text{Prob } (n > mk) < (1 - \eta^k)^m,$$

by Theorem 2. This shows that the series in (21) is dominated by a convergent geometric progression and therefore must converge.

The probability of ending on P′ in the ith step can be broken down into the usual cases, giving the equation,

$$P_i(P, P') = \sum_{j=1}^{5} p_j(P) P_{i-1}(P_j, P'), \quad i = 1, 2, \ldots.$$

Therefore

$$P_i(P, P') = \sum_{j=1}^{5} p_j(P) \sum_{i=1}^{\infty} P_{i-1}(P_j, P')$$

and

$$G(P, P') = P_0(P, P') + \sum_{j=1}^{5} p_j(P) \, G(P_j, P'),$$

with $G(Q, P') = 0$, Q on C_h. Transposing and noting that $P_0(P, P') = \delta(P, P')$, we obtain Theorem 3.

Theorem 4: The solution of problem D_h with the equation $L(V) = 0$ replaced by $L(V) = F(x, y)$, is given by

$$V(P) = W(P) - h^2 \sum_{P' \subset R_h} \frac{G(P, P') \, F(P')}{D(P')},$$

where W(P) is the solution of $L(W) = 0$ with boundary values $W(Q) = \phi(Q)$, Q on C_h.

The proof is by direct substitution into the equation $L(V) = F(P)$, using Theorem 3.

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a. Proved by different methods under less general conditions, and in one dimension, by Wald.[26, 28] The result is of importance in sequential analysis because it implies that sequential tests satisfying very weak conditions always terminate with probability one.

b. It should be noted that the function $V(x, m\lambda)$ of Section 8 with A = B = 1 gives the probability of leaving (a, b) in at most m steps, and therefore

$$\text{Prob } (n > m) = 1 - V(x, m\lambda).$$

Therefore $$V(x, km\lambda) > 1 - (1 - \eta^k)^m.$$

It is also of interest to note that $1 - V(x, m\lambda)$ is equal to the probability that the maximum and minimum of the sums

$$x + \sum_{j=1}^{K} X_j, \quad K = 1, 2, \ldots, m$$

lie within the interval $a < x < b$. The asymptotic properties of this probability as $m \to \infty$ have been investigated at some length recently by Erdös and Kac,[8] Wald[27] and Chung[4] in cases in which the length of the interval [a, b] increases with m. The fact that $V(x, m\lambda)$ satisfies equation (15) suggests a possibly fruitful approach to such problems.

It is evident from the theorem that $G(P,P')$ is analogous to the Green's function of Problem D. It is possible to extend the results discussed in Sections 8 and 10 to show that as $h \to 0$, the solution of $L(V) = F$ given above approaches the solution of Problem D with the zero on the right side of (16) replaced by $F(x,y)$.

A concept closely related to the mean number of visits to P', and one which has received a good deal of attention in the physical literature, is that of the probability of return to a point which has once been visited. Let $r(P')$ denote the probability of at least one return to P' after a visit. Let $v(P,P')$ denote the probability of reaching P' from P before first reaching the boundary. Then it is easily shown that

$$1 - r(P') = \frac{1}{G(P',P')} = \frac{v(P,P')}{G(P,P')} . \qquad (22)$$

The demonstration will be postponed to Section 15. It is to be noted that the third member of the equation is independent of P.[a]

The Mean Length of the Random Walk 12

Consider the random walk on R_h of Sections 6 and 9, and let the random variable $M(P,P')$ denote the number of visits to a point P' on R_h if the walk starts at P on R_h. The mean value of $M(P,P')$ is $G(P,P')$.

Now the total number of visits to all points of R_h, counting the start of the walk as a visit to P, is

$$\sum_{P' \subset R_h} M(P,P').$$

This is, of course, equal to n, the number of steps in the walk. The sum has only a finite number of terms, so its mean value is the sum of the mean values of the terms. Therefore

$$E(n|P) = \sum_{P' \subset R_h} G(P,P') .$$

With an eye on the statement of Theorem 4, this formula can be rewritten as follows:

$$E(n|P) = -h^2 \sum_{P' \subset R_h} \frac{G(P,P')}{D(P')} \left[\frac{-D(P')}{h^2}\right].$$

Theorem 4 then implies the following result:

Theorem 5: The mean value $E(n|P)$ as a function of P, is a solution of the problem

$$L[E(n|P)] = -\frac{D(P)}{h^2}, \quad P \text{ on } R_h, \qquad (23)$$
$$E(n|Q) = 0, \quad Q \text{ on } C_h .$$

---

a. The problem of the limiting value of $v(P,P')$ as the size of the region R increases is discussed by Polyà[21].

It is possible to extend the results discussed in Sections 8 and 10 to show that in the limit as $h \to 0$, $h^2 E(n|P)$ approaches for each P on R_h the solution of Problem D with $\phi \equiv 0$ and with the zero on the right side of (16) replaced by $-2\beta(x)$ in one dimension and by $2\beta_{12}(x,y) - 2\beta_{11}(x,y) - 2\beta_{22}(x,y)$ in two dimensions.[a]

An important special case is that in which the coefficients of $L(V)$ are constants. Suppose first that $\alpha \neq 0$, or $\alpha_1 + \alpha_2 \neq 0$. It can then be verified by direct substitution that the solution of (23) is

$$E(n|x) = \frac{D}{2h^2\alpha} [V^*(x) - x] \qquad (24)$$

in one dimension, and

$$E(n|x,y) = \frac{D}{2h^2(\alpha_1 + \alpha_2)} [V^*(x,y) - (x+y)] \qquad (25)$$

in two dimensions, where V^* is the solution of Problem D_h with boundary values

$\phi(x) = x$, or $\phi(x,y) = x + y$.[b]

If $\alpha_1 \neq 0$, then an alternative formula to (25), which is valid whether or not $\alpha_1 + \alpha_2 = 0$, is

$$E(n|x,y) = \frac{D}{2h^2\alpha_1} [V^*(x,y) - x] ,$$

where now V^* is the solution of Problem D_h with boundary values $\phi(x,y) = x$. There is, of course, a similar formula involving α_2 instead of α_1.

If $\alpha = 0$, or $\alpha_1 = \alpha_2 = 0$, it can be verified that the solutions of (23) for one and two dimensions are, respectively,

$$E(n|x) = \frac{V^{**}(x) - x^2}{h^2} , \qquad (26)$$

$$E(n|x,y) = \frac{V^{**}(x,y) - (x^2 + y^2 - xy)}{h^2} , \qquad (27)$$

where V^{**} is the solution of Problem D_h with $\phi(x) = x^2$ or $\phi(x,y) = x^2 + y^2 - xy$.

It is interesting to put (24) and (25) in another form. In the two dimensional case, suppose that all motion in the random walk takes place in directions

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a. Theorem 5 and the statement of this paragraph recently were generalized by W. Wasow to the more general random walks considered in Sections 8 and 10. Recently David[6] has proved that in one dimension, with identically distributed steps X_k whose distributions are independent of x and possess a continuous density function, $E(n|P)$ considered as a function of $Z_1 = b - a$ and $Z_2 = b - x$ satisfies a hyperbolic partial differential equation in Z_1 and Z_2.

b. The distribution of n in case the variables X_k are independent and have identical bounded discrete distributions was derived by Wald[26,28] in one dimension and Blackwell and Girschick[2] in two.

parallel to the axes; no short cuts are allowed. Then the term on the right side of each equation outside of the square bracket is the reciprocal of the mean displacement in a single step. (This has been established explicitly in one dimension in Section 8, and follows in two dimensions by inspection of the definitions of $p_1, \ldots, p_5$ given in connection with (18).) The expression "inside the square bracket" is the mean value of the total displacement S_n in the walk because V^* is the mean value of the terminal value of x or x + y. Thus, if we denote the displacement in a single step by μ, and if we let $E(S_n|P)$ be the mean total displacement if the walk starts at P, then (24) and (25) become

$$E(S_n|P) = \mu E(n|P). \tag{28}$$

A similar interpretation of (26) and (27) is possible in terms of squared displacements.

The equation (28) has been studied in one dimension in connection with determining the average sample number in sequential analysis. Wald[28] established its validity in the case in which the variables X_k have identical discrete distributions possessing moment generating functions, and Blackwell[1] proved that it holds generally if either (a) the X_k are identically distributed or (b) the X_k are all bounded and have the same mean.[a]

These results have practical significance in the solution of differential equations by random walks, inasmuch as they permit an a priori estimate to be made of $E(n|P)$ by using the fact that V^* and V^{**} must assume their maxima and minima on C_h (see Section 13 below), or by using estimates for the solutions of the differential equations satisfied in the limit. The more general derivations of (28) given by Wald and by Blackwell suggest that in practice the use of a more complicated random walk such as those considered in Sections 8 and 10 in place of the one on R_h chosen by the difference equation $L(V) = 0$ itself, might not be profitable because it would not in general reduce the mean number of steps before reaching the boundary. However, the more complicated walk might provide a better approximation to the solution of Problem D for a given value of λ.

Simple explicit formulas can be given for $E(n|P)$ in the one dimensional case in which α and β are constant and the random walk is that of Section 6. Let a' and b' be the terminal points of the walk in $x \leq a$ and $x \geq b$ respectively. By direct substitution into (23) it can be verified that if $\alpha \neq 0$, the solution is

$$E(n|x) = \frac{\beta + h\alpha}{\alpha h^2}\left[(b' - a') \cdot \frac{e^{x/h} - e^{a'/h}}{e^{b'/h} - e^{a'/h}} - x + a' \right];$$

and if $\alpha = 0$, the solution is

$$E(n|x) = \left[\frac{x - a'}{h}\right] \cdot \left[\frac{b' - x}{h}\right].$$

Thus, for example, if $a = a' = 0$, $b = b' = 50$, and if a random walk with $p = q = 1/2$ were to be set up on a lattice with a mesh constant of $h = 1$, and finally if the starting point x were chosen in the middle of the interval, then the mean number of steps would be $25 \times 25 = 625$. This would be less if $p \neq q$.

The Degree of Approximation of the Solution of the Difference Equation to that of the Differential Equation[a]. 13

Let U(P) be the solution of Problem D with $\mathcal{L}(U) = 0$ replaced by $\mathcal{L}(U) = F(P)$. The implication in the statement of Problem D is that U is defined only in $\bar{R}$. Its definition will now be extended by requiring it to assume the values $\phi(Q)$ on C_h, so that the differences in L(U) are defined for every point on R_h. A function $\epsilon(P)$ is next introduced, defined on $\bar{R}_h$ by the equation

$$L(U) = \mathcal{L}(U) - \epsilon(P).$$

If U satisfies the equation $\mathcal{L}(U) = F(P)$, then it clearly satisfies the difference equation

$$L(U) = F(P) - \epsilon(P) \text{ on } R_h.$$

Therefore, by Theorem 4,

$$U(P) = W(P) - h^2 \sum_{P' \subset R_h} \frac{G(P,P')\,F(P')}{D(P')} + h^2 \sum_{P' \subset R_h} \frac{G(P,P')\,\epsilon(P')}{D(P')},$$

where W(P) is the solution of $L(W) = 0$ with boundary values $\phi(Q)$. But the first two terms on the right constitute, by Theorem 4, the solution of $L(V) = F(P)$ with boundary values $\phi(Q)$ on C_h. That is,

$$U(P) - V(P) = h^2 \sum_{P' \subset R_h} \frac{G(P,P')\,\epsilon(P')}{D(P')},$$

and

$$|U(P) - V(P)| \leq h^2 \epsilon_{max} \sum_{P' \subset R_h} \frac{G(P,P')}{D(P')}, \tag{29}$$

where ϵ_{max} is the maximum of $\epsilon(P)$ for P on R_h. Another form of this inequality is obtained by noticing that

$$E(n|P) = \sum_{P' \subset R_h} G(P,P'):$$

$$|U(P) - V(P)| \leq h^2 \epsilon'_{max} E(n|P),$$

where ϵ'_{max} is the maximum of $\epsilon(P)/D(P)$ on R_h.

These inequalities are fundamental in studying the degree of approximation of the solution of Problem

a. The statistical writers do not seem to have noticed the connection between $E(n|P)$ and the difference equation (23).

a. Much of the material in this Section was developed independently by Wasow.

D_h to that of Problem D. They make such a study depend on estimates of the difference $\epsilon(P)$ between $\mathcal{L}(U)$ and $L(U)$, and on estimates of $E(n|P)$ or the summation in (29).

The problem of estimating $\epsilon(P)$ will be postponed for a moment.

Two general approaches to the estimation of $E(n|P)$ or of the summation in (29) are possible. The first is, so to speak, an a posteriori method; that is, the actual value of $E(n|P)$ can often be estimated statistically with sufficient accuracy for present purposes after several hundred random walks have been performed. The second method is to use mathematical a priori estimates. This will now be discussed in more detail.

In Section 12, explicit formulas were given for $E(n|P)$ in certain cases. From these, and from the fact that the values of V^* and V^{**} must lie between the maxima and minima of the boundary values of these functions, bounds for $E(n|P)$ can easily be found as was previously mentioned in Section 12. For example, in the case of formula (27),

$$E(n|x,y) \leq \frac{\max\limits_{(x',y')\text{ on }C_h} (x'^2 + y'^2 - x'y') - (x^2 + y^2 - xy)]}{h^2}.$$

The summation in (29) multiplied by h^2 is, by Theorem 4, the solution of $L(V) = -1$ with boundary values $\phi(x,y) = 0$ on C_h. Although it may be difficult to arrive at a reasonable bound for the solution of $L(V) = -1$ in general cases, nevertheless the following device is often feasible: Find a function $z(P)$ such that $L(z) \geqslant L_0 > 0$ on R_h. Then

$$h^2 \sum_{P' \subset R_h} \frac{G(P,P')}{D(P')} \leq \frac{\max\limits_{Q\text{ on }C_h} |z(Q) - z(P)|}{L_0}. \tag{30}$$

If, for instance, $\alpha_1(P) > 0$ on R_h, then a possible $z(P)$ would be simply $z(P) = z(x,y) = x$. For in this case, $L(x) = 2\alpha_1(x,y)$, and L_0 can be taken as the minimum of $2\alpha_1(P)$ on R_h. Or again, if $\alpha_1(x,y)$ is positive for $x > x_0$ and negative for $x < x_0$, then $z(P)$ can be taken as $(x - x_0)^2$. Many such functions $z(P)$ can usually be found in any given case. The goodness of the resulting estimate depends on a skillful choice of $z(P)$.

The proof of (30) is as follows: Let $Z(P)$ denote the solution of $L(Z) = 0$ with the boundary values $z(Q)$ on C_h. Then $Z^*(P) = Z(P) - z(P)$ has these properties:

(1) $Z^*(Q) = 0$, Q on C_h.

(2) $L(Z^*) = -L(z)$, where $L(z) \geqslant L_0 > 0$ on R_h.

(3) $|Z^*(P)| \leq \max\limits_{Q\text{ on }C_h} |z(Q) - z(P)|$.

The first two properties, by Theorem 4, yield the relation

$$Z^*(P) = h^2 \sum_{P' \subset R_h} \frac{G(P,P')\ .\ L(z(P'))}{D(P')}$$
$$\geqslant L_0 h^2 \sum_{P' \subset R_h} \frac{G(P,P')}{D(P')}.$$

Combining this with the third property, (30) is obtained at once.[a]

The problem of estimating ϵ_{max} in (29) will now be considered. It is known that under the conditions in the statement of Problem D in Section 9, and under the further condition that $F(P)$ has continuous first and second derivatives in a region containing R, the solution of $\mathcal{L}(U) = F(P)$ will have continuous first and second derivatives in R. Furthermore, in the one dimensional case, if the lattice is so arranged that $C_h = C$, then the solution of $\mathcal{L}(U) = F(x)$ is continuous in a closed interval containing $\bar{R}_h$ and has continuous first and second derivatives in the corresponding open interval. It also takes on the same boundary values at the same points specified in Problem D_h. In the remainder of this section it is assumed that the one-dimensional lattice has been so arranged.

The analogous situation in two and more dimensions is not so simply achieved. Without further discussion of the mathematics involved, it will be assumed henceforth that $\phi(x,y)$ and C are such that (1) ϕ is defined in the complement of R, or at least in an annular closed region surrounding R, and (2). If the definition of U, the solution of $\mathcal{L}(U) = F(P)$ in R, is extended by making U identically equal to ϕ wherever ϕ is defined exterior to R, then U has continuous first and second derivatives in a region containing R (and, therefore, containing $\bar{R}_h$ for all h sufficiently small). The statement of Problem D_h for $L(V) = F(P)$ given in Section 9 and Theorem 4 now implies that U and V coincide on C_h. Both this and the continuity of the derivatives of U in a region containing $\bar{R}_h$ are essential for what follows.[b]

The demonstration will be given in two dimensions; specialization to one dimension is immediate. By an application of Taylor's Theorem with remainder,

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a. The exposition in this paragraph and the preceding one follows closely a discussion given by Wasow in a letter to the author.

b. It would be possible, of course, to redefine $\bar{R}_h$ so that it lies entirely in R, and then to extend the definition of ϕ into the interior of R appropriately. The procedure in the text of having C_h lie outside R has been followed merely for convenience of exposition in indicating the relationship with Petrowsky's theorem.

it is easily shown that, for example,

$$\Delta_{xx} U = \frac{\partial^2 U}{\partial x^2} - \epsilon_{11}(P) , \quad P \text{ on } R_h ,$$

where $|\epsilon_{11}|$ is less than or equal to the maximum absolute deviation of $\frac{\partial^2 U}{\partial x^2}$ on the line segment $\overline{P_3P_1}$ from its value at P. Similar expressions for the other differences of U can be derived. Operating now on U with L, it is found that the quantity

$$\epsilon(P) = \mathcal{L}(U) - L(U)$$

can be written as follows:

$$\epsilon(P) = \beta_{11}\epsilon_{11} + 2\beta_{12}\epsilon_{12} + \beta_{22}\epsilon_{22} + 2\alpha_1\epsilon_{10} + 2\alpha_2\epsilon_{01} ,$$

where ϵ_{11}, ϵ_{12}, etc., are functions of P whose respective bounds for each individual P on R_h are listed in Table I. The abbreviation "max. dev." in

Function	Bound for absolute value
ϵ_{11}	max. dev. $\frac{\partial^2 U}{\partial x^2}$ on $\overline{P_3P_1}$
ϵ_{12}	$\frac{1}{2}$(max. osc. $\frac{\partial^2 U}{\partial x^2}$ in square $PP_1P_5P_2$) + $\frac{1}{2}$(max. osc. $\frac{\partial^2 U}{\partial y^2}$ in same square) + max. dev. $\frac{\partial^2 U}{\partial x \partial y}$ in same square.
ϵ_{22}	max. dev. $\frac{\partial^2 U}{\partial y^2}$ on $\overline{P_4P_2}$
ϵ_{10}	max. dev. $\frac{\partial U}{\partial x}$ on $\overline{P\,P_1}$
ϵ_{01}	max. dev. $\frac{\partial U}{\partial y}$ on $\overline{P\,P_2}$

Table I

the second column of this table means the maximum possible absolute deviation of the value of the designated function from its value at P, and "max. ocs." means the maximum oscillation of the functions; i.e., for a function f(P) the maximum of $|f(P') - f(P'')|$ for P' and P'' on the designated set of points.

An upper bound for $\epsilon(P)$ on R_h, and therefore for ϵ_{max}, can be obtained by replacing the ϵ's in the expression for $\epsilon(P)$ by the overall maxima on R_h of their respective bounds as given in the table, and replacing the coefficients in $\epsilon(P)$ by their maxima on $\bar{R}$.

It is now an obvious consequence of the assumptions on the continuity of the derivatives of U that

$$\lim_{h \to o} \epsilon_{max} = 0$$

uniformly on $\bar{R}$. If the existence and uniform continuity of the second derivatives of U in some region containing R (or merely in the interval $a \leq x \leq b$ in the one dimensional case) is considered an acceptable assumption, and if an a priori estimate of $E(n|P)$ or of the summation in (29) is at hand, which, when multiplied by h^2, is bounded as h approaches zero, then the discussion in this section may be considered to provide a proof that as h approaches zero, the solution of Problem D_h approaches that of Problem D. These assumptions are certainly not all fulfilled in certain cases admitted in the original statement of Problem D, as, for instance, when $\phi(x, y)$ is only piecewise continuous. Limitations of space prevent further discussion of such matters, and the material in this section is presented accordingly from the point of view of providing a basis for estimating the degree of approximation of V(P) to U(P) in special cases, rather than from that of providing a definitive asymptotic treatment of the solution of Problem D_h. Such a treatment is, in fact, furnished by Petrowsky's work.[20]

The Dispersion of the Statistical Estimate of V(P) and the Number of Samples Necessary to Achieve a Given Accuracy. 14

The Monte Carlo solution of Problem D consists in (1) assuming that for a fine enough mesh, or a small enough value of λ, the theoretical mean value V(P) gives a satisfactory approximation to the solution of the differential equation, and (2) estimating V(P) by one of the usual estimators employed in statistics. The most common estimator is the simple average, or arithmetic mean, of the N tallies obtained in N random walks, and we shall consider here only this estimator.

The argument which follows applies to the general random walks considered in Sections 8 and 10.

Let $\phi_1, \phi_2, \ldots, \phi_N$ be independent, identically distributed, random variables representing the tally after each of the N walks, which are all assumed to start from the fixed point P on R_h. Then for j = 1, . . . , N,[a]

$$E(\phi_j) = V(P) ,$$

---

a. Throughout this paper, the symbol E(X) will be used for the mean value of the random variable X.

and the variance[a] $\sigma^2(\phi_j)$ is given by

$$\sigma^2(\phi_j) = E[\phi_j - V(P)]^2$$
$$\leq E[\phi_j - M]^2 \leq \overline{\text{bound}}[\phi(Q) - M]^2 \ , \tag{31}$$

where Q in the last member is any point exterior to R on which the walk can end with non-zero probability, and M is any real number whatsoever. The inequality follows from the fact that the second moment of any distribution is least about the arithmetic mean.

The statistical estimate of V(P) will be, a priori,

$$\bar{\phi} = \frac{\phi_1 + \phi_2 + \ldots + \phi_N}{N} \ .$$

Its mean value is, of course, V(P), and its variance is given by

$$\sigma^2(\bar{\phi}) = \frac{\sigma^2(\phi_j)}{N} \leq \frac{\overline{\text{bound}}[\phi(Q) - M]^2}{N} \ . \tag{32}$$

The estimator ϕ will be nearly normally distributed for $N \geq 100$, by the Central Limit Theorem of probability theory, so

$$\text{Prob}\ \{|\bar{\phi} - V(P)| > 2\sigma(\phi)\}$$

will be about 0.05. If this is considered a small enough margin of error, and if it is desired to compute V(P) to, say, m decimal places, then an upper bound for the necessary number of samples can be determined as follows: Choose M so as to make the last member of (32) as small as possible, and let

$$2\frac{\overline{\text{bound}}\ |\phi(Q) - M|}{\sqrt{N}} = \frac{5}{10^{m+1}} \ .$$

This yields the formula

$$N = 16 \cdot 10^{2m}\ \overline{\text{bound}}[\phi(Q) - M]^2 \ . \tag{33}$$

It is worthwhile writing out the distribution of the endpoint of the walk and the formulas for $E(\phi_j)$ and $\sigma^2(\phi_j)$ in the case of the random walk on R_h discussed in Sections 6 and 9. This can be done with the aid of the function $v(P, P')$ introduced at the end of Section 11, but with P' now on C_h. In that case, $v(P,P')$ is the solution of $L(V) = 0$ with $\phi(Q) = 0$, Q on C_h, $Q \neq P'$, and $\phi(P') = 1$. In other words, $v(P,P')$ is the probability of reaching the particular point P' before reaching any other point exterior to R.

a. The variance of a distribution or of a random variable is defined to be the second moment of the distribution about the mean value. The square root of the variance is called the standard deviation, or standard error. The notation $\sigma^2(X)$ will always be used for the variance of the random variable X.

The probability distribution of the endpoint of the random walk can now be written out as shown in Table II, in which $Q_1, Q_2, \ldots$ represent the individual points of C_h.

Position of endpoint	Q_1	Q_2	Q_3	...
Probabilities	$v(P,Q_1)$	$v(P,Q_2)$	$v(P,Q_3)$	...

Table II

Then

$$E(\phi_j) = \sum_i \phi(Q_i)\ v(P,Q_i) = V(P)$$

and (31) becomes

$$\sigma^2(\phi_j) = \sum_i [\phi(Q_i) - V(P)]^2\ v(P,Q_i)$$
$$\leq \overline{\text{bound}}_i\ [\bar{\phi}(Q_i) - M]^2 . \tag{34}$$

Of course (32) remains unchanged.

In one dimension, these formulas yield the following expression for the variance of $\bar{\phi}$:

$$\sigma^2(\bar{\phi}) = (B - A)^2 \frac{v(x,b')\ (1 - v(x,b'))}{N} \ . \tag{35}$$

This is greatest when $v(x,b') = \frac{1}{2}$, in which case we get the upper bound

$$\sigma^2(\bar{\phi}) \leq \frac{(B - A)^2}{4N} \ .$$

This is exactly the value that would have been obtained by letting $M = (A + B)/2$ in (32), so it follows that with proper choice of M, the bound given by (32) under certain very special circumstances is the best one possible. Formula (33) now becomes

$$N = 4 \cdot 10^{2m} (B - A)^2 \ . \tag{36}$$

Unless the boundary values are nearly constant, or unless only low accuracy is needed, it is obvious that the number of samples required as estimated by (33) or (36) will be appallingly large. Less pessimistic estimates can be obtained by working directly with the first two members of (31), or of (34), or with (35), using knowledge as to the approximate value of $v(P, Q)$. The technique of sequential analysis[28] can also be used profitably to cut down the sample size, the maximum possible saving over techniques requiring a fixed number of samples is of the order of about 50%. Other techniques possibly leading to a much greater reduction in the overall amount of computing will be considered in Section 16.

The Dispersion of the Statistical Estimate of $G(P,P')$ 15

We now calculate the variance of the estimate of and the variance[a] $\sigma^2(\phi_j)$ is given by

$$\begin{aligned}\sigma^2(\phi_j) &= E[\phi_j - V(P)]^2 \\ &\leq E[\phi_j - M]^2 \leq \overline{\text{bound}}[\phi(Q) - M]^2 \quad ,\end{aligned} \tag{31}$$

where Q in the last member is any point exterior to R on which the walk can end with non-zero probability, and M is any real number whatsoever. The inequality follows from the fact that the second moment of any distribution is least about the arithmetic mean.

The statistical estimate of V(P) will be, a priori,

$$\bar{\phi} = \frac{\phi_1 + \phi_2 + \ldots + \phi_N}{N} \quad .$$

Its mean value is, of course, V(P), and its variance is given by

$$\sigma^2(\bar{\phi}) = \frac{\sigma^2(\phi_j)}{N} \leq \frac{\overline{\text{bound}}[\phi(Q) - M]^2}{N} \quad . \tag{32}$$

The estimator ϕ will be nearly normally distributed for $N \geq 100$, by the Central Limit Theorem of probability theory, so

$$\text{Prob } \{|\bar{\phi} - V(P)| > 2\sigma(\phi)\}$$

will be about 0.05. If this is considered a small enough margin of error, and if it is desired to compute V(P) to, say, m decimal places, then an upper bound for the necessary number of samples can be determined as follows: Choose M so as to make the last member of (32) as small as possible, and let

$$2\,\frac{\overline{\text{bound}}\ |\phi(Q) - M|}{\sqrt{N}} = \frac{5}{10^{m+1}} \quad .$$

This yields the formula

$$N = 16 \cdot 10^{2m}\ \overline{\text{bound}}[\phi(Q) - M]^2 \ . \tag{33}$$

It is worthwhile writing out the distribution of the endpoint of the walk and the formulas for $E(\phi_j)$ and $\sigma^2(\phi_j)$ in the case of the random walk on R_h discussed in Sections 6 and 9. This can be done with the aid of the function $v(P, P')$ introduced at the end of Section 11, but with P' now on C_h. In that case, $v(P,P')$ is the solution of $L(V) = 0$ with $\phi(Q) = 0$, Q on C_h, $Q \neq P'$, and $\phi(P') = 1$. In other words, $v(P,P')$ is the probability of reaching the particular point P' before reaching any other point exterior to R.

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a. The variance of a distribution or of a random variable is defined to be the second moment of the distribution about the mean value. The square root of the variance is called the standard deviation, or standard error. The notation $\sigma^2(X)$ will always be used for the variance of the random variable X.

The probability distribution of the endpoint of the random walk can now be written out as shown in Table II, in which $Q_1, Q_2, \ldots$ represent the individual points of C_h.

Position of endpoint	Q_1	Q_2	Q_3	...
Probabilities	$v(P,Q_1)$	$v(P,Q_2)$	$v(P,Q_3)$	...

Table II

Then

$$E(\phi_j) = \sum_i \phi(Q_i)\, v(P,Q_i) = V(P)$$

and (31) becomes

$$\begin{aligned}\sigma^2(\phi_j) &= \sum_i [\phi(Q_i) - V(P)]^2\, v(P,Q_i) \\ &\leq \overline{\text{bound}}_i\ [\bar{\phi}(Q_i) - M]^2 .\end{aligned} \tag{34}$$

Of course (32) remains unchanged.

In one dimension, these formulas yield the following expression for the variance of $\bar{\phi}$:

$$\sigma^2(\bar{\phi}) = (B - A)^2\, \frac{v(x,b')\,(1 - v(x,b'))}{N} \ . \tag{35}$$

This is greatest when $v(x,b') = \frac{1}{2}$, in which case we get the upper bound

$$\sigma^2(\bar{\phi}) \leq \frac{(B - A)^2}{4N} \quad .$$

This is exactly the value that would have been obtained by letting $M = (A + B)/2$ in (32), so it follows that with proper choice of M, the bound given by (32) under certain very special circumstances is the best one possible. Formula (33) now becomes

$$N = 4 \cdot 10^{2m} (B - A)^2 \ . \tag{36}$$

Unless the boundary values are nearly constant, or unless only low accuracy is needed, it is obvious that the number of samples required as estimated by (33) or (36) will be appallingly large. Less pessimistic estimates can be obtained by working directly with the first two members of (31), or of (34), or with (35), using knowledge as to the approximate value of $v(P,Q)$. The technique of sequential analysis[28] can also be used profitably to cut down the sample size, the maximum possible saving over techniques requiring a fixed number of samples is of the order of about 50%. Other techniques possibly leading to a much greater reduction in the overall amount of computing will be considered in Section 16.

The Dispersion of the Statistical Estimate of $G(P,P')$ 15

We now calculate the variance of the estimate of

Instead of $p_1, p_2, \ldots, p_5$, a new set of transition probabilities $p_1{*}(P), p_2{*}(P), \ldots, p_5{*}(P)$ is chosen at each point P on R_h such that no one is zero and

$$\sum_1^5 p_j{*}(P) = 1.$$

Equation (18) is replaced by the equivalent one:

$$V(P) = \sum_{j=1}^{5} \left[p_j{*}(P) \cdot \frac{p_j(P)}{p_j{*}(P)} \ V(P_j) \right] . \qquad (38)$$

This is interpreted in probability language as follows: A particle starts at the point P on R_h and performs the random walk described in Section 9 but with the transition probabilities $p_1{*}, \ldots, p_5{*}$. If the first step is to $P_j, j = 1, 2, \ldots, 5$, the weight $p_j(P)/p_j{*}(P)$ is attached to the particle. Then if the next step is to P_i', $i = 1, 2, \ldots, 5$, this weight is multiplied by $p_i(P_j)/p_i{*}(P_j)$, and so on, until a point Q of C_h is reached. The cumulated product of weights is then multiplied by $\phi(Q)$ and tallied as the final score for the walk.

It is easy to see, in various ways, that V(P) as previously defined must also be the mean value of the final tally. For one thing, the mean value of the tally clearly satisfies (38) with the proper boundary values. Another way to look at the matter is to consider the contribution to V(P) arising from a particular path from P^0 in R_h to Q in C_h. We suppose that this path proceeds via the particular points P^k, $k = 1, 2, \ldots$. The probability that the particle took this path is

$$p_{j_0}{*}(P^0) \cdot p_{j_1}{*}(P^1) \cdot p_{j_2}{*}(P^2) \ldots, \qquad (39)$$

where each subscript j_k is some one of the five numbers 1, 2, . . . , 5. The tally at the end of the path is

$$\phi(Q) \ \frac{p_{j_0}(P^0) \cdot p_{j_1}(P^1) \ldots}{p_{j_0}{*}(P^0) \cdot p_{j_1}{*}(P^1) \ldots} . \qquad (40)$$

The mean value of the tally for this particular path is then simply the product of (39) and (40), or

$$\phi(Q) \cdot \prod_k p_{j_k}(P^k) ,$$

but this is exactly the mean value which would have been obtained from this path if the walk had proceeded according to the transition probabilities $p_i(P)$ instead of $p_i{*}(P)$.

In applying the method to estimate $V(P^0)$, the estimator will be the average of N tallies obtained in N walks. Let z be a random variable whose values are all of the various possible tallies and whose distribution is given by the corresponding probabilities of the tallies. Then

$$E(z) = V(P^0) = \sum_{Q \subset C_h} \phi(Q) \sum \prod_k \left[\frac{p_{j_k}(P^k)}{p_{j_k}{*}(P^k)} \cdot p_{j_k}{*}(P^k) \right]$$

and

$$\sigma^2(z) = -[V(P^0)]^2 + \sum_{Q \subset C_h} [\phi(Q)]^2 \sum \prod_k \left\{ \left[\frac{p_{j_k}(P^k)}{p_{j_k}{*}(P^k)} \right]^2 p_{j_k}{*}(P^k) \right\} , \qquad (41)$$

where the inner summation is extended over all possible paths leading from P^0 to Q. The estimate of V(P) based on z, will be

$$\bar{z} = \frac{z_1 + z_2 + \ldots + z_N}{N}$$

where $z_1, z_2, \ldots$ denote separate independent observations on z. The variance of the estimator is

$$\sigma^2(\bar{z}) = \sigma^2(z)/N.$$

Formula (41) has two interesting consequences. In the first place, it is clear that if some one of the p*'s is made to approach zero and if the corresponding p is not zero, then $\sigma^2(z)$ becomes infinite. Therefore, it is evidently possible to make a choice of the p*'s which will make the standard error of the estimate of V(P) very much more unfavorable than it would have been if the natural probabilities $p_1, \ldots, p_5$ had been used.

In the second place, a good choice of the p*'s may result in a reduction in the standard error. Indeed, if there is a priori knowledge concerning the solution V(P), the variance $\sigma^2(\bar{z})$ can be very greatly reduced. Consider the extreme case in which V(P) is already known on R_h. Suppose V(P) is always positive on R_h. (This can always be brought about by merely adding a constant to the boundary value $\phi(Q)$.) Now let

$$p_j{*}(P) = p_j(P) \cdot \frac{V(P_j)}{V(P)} , \quad j = 1, \ldots, 5 .$$

In view of (18), $\sum_1^5 p_j{*}(P) = 1$.

It is now quite easily seen by substitution that the double summation in (41) has the value

$$\sum_{Q \subset C_h} [\phi(Q)]^2 \left[\frac{V(P^0)}{\phi(Q)} \right]^2 \sum \prod_k p_{j_k}(P^k) = [V(P^0)]^2 ,$$

and so $\sigma^2(z) = \sigma^2(\bar{z}) = 0$.

The existence of transition distributions leading to zero standard errors of estimates was pointed out by Kahn[11] in connection with mechanical quadrature and the integration of certain integro-differential equations of diffusion theory.[a]

Of course, in practice, V(P) will not be known, but a suitable approximation can be used instead, perhaps based on physical theory or intuition. The author has seen cases in which the number of samples required to achieve a given accuracy has been reduced by a factor of 1/100 by this device.

Another way of using the technique of alteration of transition probabilities consists in making the new probabilities constant and, in particular, all equal. In the two-dimensional case, with $\beta_{12} \neq 0$, the individual values, if equal, would be 1/5. This technique might be convenient for certain types of computing machines as the handling of the random numbers is simpler. The effect on $\sigma^2(\bar{z})$ might be unfortunate however, and, if possible, an analysis of (41) should be made in advance.

The non-homogeneous equation $L(V) = F(x, y)$, when rearranged in the form (18), becomes

$$V(P) = \sum_{j=1}^{5} p_j(P)\, V(P_j) - \frac{h^2}{D(P)}\, F(P) .$$

From the meaning of $G(P, P')$, the direct statistical estimate of the second term in the solution as given in Theorem 4 is the average of all the values of $h^2\, F(P)/D(P)$ observed at each resting place in R_h, counting the value at the starting point into the average, and using the transition probabilities $p_1(P)$, . . . , $p_5(P)$. But it can be seen by repeating the argument used in connection with (39) and (40) that $G(P, P')$ can also be considered as the mean value of the various weights which the particle has attached to it on its various visits to P' (with a weight of unity added on if $P' = P$). Therefore, it is possible to use the following procedure for estimating the solution of the non-homogeneous equation: Using the altered transition probabilities $p_1^*, \ldots, p_5^*$, calculate at each interior resting point P' of the walk the proper weight $w(P')$ as previously directed, and also calculate the quantity $T = -\, w(P')h^2\, F(P')/D(P')$ and add it to the cumulative sum of all previous quantities T so calculated in the course of the walk. The cumulative sum is to start with the term

$$-h^2\, F(P)/D(P).$$

The final tally is the product of $\phi(Q)$ into the cumulated product-weight at arrival, plus the cumulative sum of the values of T upon arrival. (Zero is added to T at Q.)

Bookkeeping Procedures 17

It is advantageous in many cases to extract as much information from a single random walk as possible. This matter has already been touched upon in Section 15, where it was suggested that if $G(P, P')$ were to be calculated for a fixed P, and for every point P' on R_h, then a natural procedure might be to observe all of the values of $n(P, P')$ for each P' on R_h in each individual random walk, and average these parallel values over N walks. This entails keeping a complete record of the sequence of events in each random walk. Such a diary may become quite bulky, and the decision as to how complete it should be will be affected by the storage facilities of the computing machinery to be used.

The bookkeeping approach will probably often be convenient when the value of the solution V(P) of Problem D_h is to be estimated at several points of R_h. Two procedures now suggest themselves. Suppose that the solution is to be obtained at the points P and P', that N random walks on R_h are started at P, and that N' of these arrive at P' before reaching C_h. The first procedure consists in treating each one of these N' random walks after the first visit to P' as a single new random walk starting at P'. The mean number of walks thus gained for P' is $Nv(P, P')$ where $v(P, P')$ in the function used in Section 15 and elsewhere. In one dimension, at least, this will usually be a substantial fraction of N.

The second procedure consists in regarding each visit to P' as starting a new random walk at P'. Thus in the notation of Section 15, each random walk starting from P generates $n(P, P')$ new random walks at P'. This second procedure, clearly, also has significance as providing a new estimation procedure at P itself; this is the case in which P' coincides with P.

However, Wasow has found that the second procedure is disadvantageous, as it increases the dispersion of the statistical estimate of $V(P')$. His demonstration (which was communicated to the author informally in a letter) runs essentially as follows:

Consider the first procedure for a moment. As in Section 14, let $\phi_1, \phi_2, \ldots, \phi_{N'}$ be identically

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a. See also the paper by Kahn and Harris.[19] The method is called that of "importance sampling" by these writers. Dr. W. Edwards Deming pointed out to the author that the method has been known for many years in various forms to sampling experts working in the social sciences, who have sometimes called it "sampling with probability in proportion to size". (See Ref. 7, pp. 92-93.)

distributed independent random variables representing the tally after each of the N' walks started from P', so $E(\phi_j) = V(P')$. The estimate of $V(P')$ in this procedure is, a priori,

$$\bar{\phi} = \frac{\sum_1^{N'} \phi_j}{N'} .$$

Although if the walks were started at N, then N' must be a random variable, it will, nevertheless, now be assumed that N' is fixed; that is, we shall study only the conditional distribution of $\bar{\phi}$, given N'. It is assumed naturally that $N' > 0$. (Of course if P' coincides with P, then $N' = N$.)

As before, we denote the variance of $\bar{\phi}$ in the first procedure by

$$\sigma^2(\bar{\phi}) = \sigma^2(\phi_j)/N' .$$

In the second procedure, the estimate of $V(P')$ corresponding to $\bar{\phi}$ is, a priori,

$$\phi^* = \frac{\sum_1^{N'} n_j \phi_j}{\sum_1^{N'} n_j}$$

where $n_j = n_j(P,P')$, is a random variable representing the number of visits to P' in the jth random walk, under the hypothesis that there is at least one such visit in that walk. (The distribution of $n_j(P,P')$ is given by Table III as it stands if $P = P'$, and if $P \neq P'$ it is given by Table III but with the value 0 deleted and the probabilities all divided by $1 - v$.) The $2N'$ variables ϕ_j and n_j are all independent.

The function ϕ^* is probabilistically more complicated than $\bar{\phi}$, as it contains random variables in both numerator and denominator. Its mean value is, nevertheless, still $V(P)$; it may be calculated in this way:

$$\begin{aligned} E(\phi^*) &= E_{n_j}\left[E_{\phi_j}(\phi^*|n_j)\right] \\ &= E_{n_j}\left[\frac{\sum_1^{N'} n_j V(P')}{\sum_1^{N'} n_j}\right] = V(P')\, E_{n_j}(1) \\ &= V(P') \quad , \end{aligned}$$

where E_{n_j} denotes the unconditional mean value with respect to the variables $n_1, n_2, \ldots, n_{N'}$, and $E_{\phi_j}(\phi^*|n_j)$ signifies the conditional mean value of ϕ^* with respect to $\phi_1, \phi_2, \ldots, n_{N'}$, for fixed values of the n_j.

In the same notation,

$$\begin{aligned} \sigma^2(\phi^*) &= E[\phi^* - V(P')]^2 \\ &= E_{n_j}\left\{E_{\phi_j}\left[(\phi^* - V(P'))^2 | n_j\right]\right\} \\ &= E_{n_j}\left\{\frac{\sum_1^{N'} n_j^2 \sigma^2(\phi_j)}{\left(\sum_1^{N'} n_j\right)^2}\right\} \\ &= \sigma^2(\phi_j)\, E_{n_j}\left\{\frac{\sum_1^{N'} n_j^2}{\left(\sum_1^{N'} n_j\right)^2}\right\} . \end{aligned} \tag{42}$$

But Schwarz's inequality states that

$$\left(\sum_1^{N'} 1 \cdot n_j\right)^2 \leqslant \sum_1^{N'} 1^2 \sum_1^{N'} n_j^2 = N' \sum_1^{N'} n_j^2 , \tag{43}$$

with the equality holding only if $n_1 = n_2 = \ldots = n_{N'}$. The joint distribution of $n_1, n_2, \ldots, n_{N'}$ can be concentrated at a single point in N'-space only in the trivial case in which there is only one lattice point on R_h. Disregarding this case, and substituting (43) into (42), we obtain the strong inequality

$$\sigma^2(\phi^*) > \frac{\sigma^2(\phi_j)}{N'} = \sigma^2(\bar{\phi}) ,$$

which was to be proved.

Wasow has further pointed out that an upper bound for $\sigma^2(\phi^*)$ now lies close at hand. For

$$\sum_1^{N'} n_j \geqslant N',$$

and the n_j are identically distributed; therefore, by the computation used to find $\sigma^2[n(P,P')]$ in Section 15,

$$\sigma^2(\phi^*) \leqslant \frac{\sigma^2(\phi_j)}{N'} E_{n_1}(n_1^2)$$

$$= \begin{cases} \dfrac{\sigma^2(\phi_j)}{N'} \cdot \dfrac{v}{1-v} \dfrac{1+r}{(1-r)^2} , & P \neq P' , \\[2ex] \dfrac{\sigma^2(\phi_j)}{N} \cdot \dfrac{1+r}{(1-r)^2} , & P = P' . \end{cases}$$

It is clear that bookkeeping procedures of the type discussed in this section can be applied to the sampling methods discussed in Section 16. In this case, the bookkeeping will involve calculating in parallel a number of scores.

Further Problems 18

The theory presented in the paper is relatively complete for the equation L(V) = F(P) in bounded regions. The eigenvalue case is being explored actively by various research workers as this is being written.

There are, however, a number of questions in connection with the problems treated in this paper which have not as yet been adequately answered. One of these is that of the degree of approximation of V(P) for a given h or λ to the solution of Problem D. The demonstration given in Section 13 of the present paper depends upon finding a suitable a priori estimate for the derivatives of the solution of problem D in terms of the given boundary value function ϕ and of C. Just how this is to be done in complicated cases is not at all clear. In this connection, furthermore, it would be interesting to obtain a useful bound for E(n|P), or for the right side of (29) valid for the most general operator L. Neither the proof of Petrowsky's Theorem, nor the standard treatments of Problem D_h [5,10,15] throw any light on this matter.

Another open problem consists of the extension of the methods of this paper to boundary value problems involving differential equations of order higher than two, and to non-linear differential equations. Further open problems are encountered when unbounded regions are considered. A considerable amount of additional research probably will be needed to give fully satisfactory answers to these questions.

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DISCUSSION

MR. ISAACSON: It occurs to me that the elliptic equation may be regarded as an equation representing the steady state solution which might be approached by the solution of the parabolic equation, in which case some of the methods which were described by Dr. Thomas might be applicable and might yield an estimate of the accuracy of your probabilistic approach to the problem. Have you considered that view of the problem, rather than the random walk method?

DR. CURTISS: Yes. The connection between the elliptic differential equation and the parabolic differential equation is this: for elliptic equations, n is a random variable; whereas for parabolic equations there is a one-sided random walk (always going forward) as far as n is concerned. You introduce a time variable which essentially varies as the reciprocal of n. The moment that has been done, it becomes a parabolic differential equation, as Courant and Friedrichs explicitly state in a footnote. The background of exactly your comment is contained in a rather extensive footnote on the pages in which this equation is treated.

CHAIRMAN TAUB: An alternative to your method for solving the parabolic equation utilizes the Khintchine theorem, which holds for parabolic differential equations as well, and this method for solving parabolic equations, should be compared to those of the first by Dr. Thomas.

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The Institute for Numerical Analysis of the National Bureau of Standards

J. H. Curtiss, Chief
National Applied Mathematics Laboratories, NBS

Dr. Curtiss discusses the program of the Institute for Numerical Analysis established for basic research and training in the types of mathematics which are pertinent to the efficient exploitation and further development of high-speed automatic digital computing equipment. He has been head of the National Applied Mathematics Laboratories, of which the Institute is one of four branches, since NAML was established in 1947. Prior to this Dr. Curtiss was with the Bureau of Ships. He has also served on the faculties of Cornell, Harvard, and Johns Hopkins Universities and is well known as a consultant on problems in mathematical statistics.

Suppose that a research worker suddenly found that he had to solve 100 linear equations in 100 unknowns, and that he had access to a large automatic digital computing machine with whose coding and operation he was familiar. How should he set about preparing the problem for the machine?

This situation is by no means unrealistic. Large sets of simultaneous equations are encountered in many data-reduction problems (e.g. in geodetic triangulation problems), in mathematical treatments of logistics and economics, and in engineering and applied physics. In fact, many numerical problems in applied mathematics, such as the numerical solution of integral and differential equations, can be reduced in one way or another to solving large sets of simultaneous linear equations, although in individual cases this may not be the optimal procedure.

It may look easy. Any high school student knows how to solve simultaneous linear equations. For example, why not write the various solutions down directly as the quotients of pairs of determinants and make the machine evaluate the determinants directly from their definition?

This is a beautiful theoretical solution; but each determinant has 100 rows and 100 columns. By definition, the value of each determinant is the algebraic sum of $100 \times 99 \times 98 \times \ldots \times 3 \times 2 \times 1$ signed products, each consisting of 100 factors chosen from among the elements of the determinant so that each row and each column is represented. On this basis, for each determinant over 10^{157} products of 100 numbers each would have to be formed and added together algebraically. If a multiplication could be done in 100 microseconds on the automatic computing machine, and an addition in 10 microseconds, and if an infinitely extensive high-speed internal memory could be assumed, then each determinant would be evaluated in something over 10^{147} years of machine time! And if all the solutions to the equations are wanted, 101 of these determinants will have to be evaluated. It looks as if the machine will be busy for some time to come.

So this method turns out to be ridiculous for a problem of this size. By use of efficient methods, each determinant can be evaluated in something like 3×10^5 multiplications and a comparable number of algebraic additions. For the 101 determinants required, the straight computing time would be something like one hour. Another 3 to 6 hours must be allowed for internal logical operations like changing the instructions within the machine. However, the assumption of unlimited memory is pure nonsense; the really high-speed (i.e., readily accessible) memories in the current crop of machines are limited to about 1000 numbers of 10 to 12 decimal digits each. To carry out the procedure it would be necessary constantly to bring data in and out of the high-speed memory, a process so slow that, by comparison, computing time can be ignored. Finally, the twin problems of significant digits and round-off errors are so serious as to make it quite unlikely that any ordinary procedure suitable for 5 or 6 equations would even roughly approximate the answer to our problem with 100 equations.

What about other methods? There is certainly no dearth of them to choose from. Dr. George E. Forsythe[1] has been working on a classification and bibliography in which he distinguishes two main categories: direct methods, like the method of determinants referred to above, and iterative methods, in which the solution is arrived at by successively closer approximations. Rapid recapitulation of some of the short names of the methods will suggest the variety of approaches available:

Direct Category: Chio and Aitken determinant methods; Bingham, R. Schmidt, Lanczos,[1] and Frame characteristic equation methods;

[1] Member of the staff of the NBS Institute for Numerical Analysis.

Gauss elimination method and many abbreviations thereof; Cholesky (or square-root) and "escalator" triangularization methods; Gram-Schmidt, Fox-Huskey[1]-Wilkinson, and Bodewig orthogonalization methods; block elimination methods; "Below-the-line" device, R. A. Fisher technique, and other special methods.

Iterative category: about a dozen distinct variants of the Wittmeyer type (some of them associated with names like Gauss, Seidel, Jacobi, and von Mises); a number of versions of the least-squares type, including an entire subclass first characterized as such by Rosser[2] and Hestenes[1]; the subclass of gradient methods with several variants; iterated elimination methods; and "relaxation" methods.

Then too, there is a separate category of miscellaneous special methods, such as "Monte-Carlo" methods (i.e., sampling methods based on probability theory), and a method of H. Lewy[3] for solving equations when the solution is known to be in integers.

This catalogue is quite probably incomplete, but even as it stands, it presents a truly bewildering problem of selection. D. R. Hartree[2] (1) says "It is probably the case that there is no one best method for the evaluation of the solution of a set of simultaneous linear algebraic equations, but that the best method in any particular case depends on the structure of the set of equations concerned. This is certainly true of methods for treating such equations without the use of an automatic machine, and may well still be true when such assistance is available. For example, it may be that most of the coefficients are non-zero, and are not small integers and moreover are known only approximately, either because they are derived from measurements which are subject to experimental error or because they are results of previous calculations and are subject to rounding-off errors; or it may be that each equation involves only a few of the variables, and these with coefficients which are small integers and are known to be exact. It is quite likely the most appropriate methods in the two cases will be different."

But how should the choice be made? Somehow, somewhere, guide lines must be laid down which will permit intelligent selection of methods, with due regard to the peculiarities of a given system of equations and to the specifications of available computing equipment. The literature is full of special methods, but the overall guide book has not yet been written. In fact, there seem to be many cases even now in which no known method gives feasible solutions. Moreover, relatively little is known about how round-off errors and other errors inherent in computational work pile up in the many methods listed above when they are applied to really large sets of equations. Perhaps the only way to resolve that particular sort of question will be by extensive arithmetical experiments on high-speed automatic machines.

Here is an important type of problem, then, which recurs over and over again in applied mathematics. A lot is known about it, but not nearly enough for the full and effective exploitation of automatic digital computing machinery, and what is already known needs to be pulled together in a usable form. In other words, this area of numerical analysis is badly needs both background research and foreground research, together with laboratory experimentation, to maximize the nation's return from its multimillion dollar investment in computing machinery.

It is to work on such problems that the Institute for Numerical Analysis of the National Bureau of Standards was established in 1947 with the support of the Office of Naval Research and the cooperation of the University of California. The Institute is located on the campus of the University of California at Los Angeles, and maintains close liaison with the University. Its permanent staff numbers about 70 and includes some 15 scientists (mathematicians and theoretical physicists) at the post-Ph.D. level. In addition, a number of scientists have worked at the Institute at one time or another on temporary appointments. Some of the outstanding mathematicians in the world have been associated for various terms with the Institute, and the permanent research staff compares in strength and productivity with the stronger academic mathematics departments.

The Institute is one of the four branches of the National Applied Mathematics Laboratories (NAML) of the National Bureau of Standards, which ONR played a fundamental role in establishing. The story of the Institute has been set forth at some length in a readily accessible article in Science (2), so only a brief outline is necessary here.

In 1945 a study of Navy Department computing requirements by the Office of Research and Invention, now ONR, resulted in a memorandum prepared by Lt. Comdr. James H. Wakelin and circulated within the Naval establishments. It in effect recommended the establishment, with Navy participation, of a national interagency computing center which would develop and use large-scale automatic machines. Early in 1946, Rear Admiral H. G. Bowen, then Chief of Naval Research, suggested to Dr. E. U. Condon, Director of the National Bureau of

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[2] Former Director of Research of the NBS Institute for Numerical Analysis.

[3] Consultant of the NBS Institute for Numerical Analysis.

Standards, that ONR and NBS should jointly undertake to establish such a facility. It was eventually agreed that NBS should be solely responsible for administration of the center, and also that ONR's participation was conditional upon commitments from other agencies to support the new activity.

It was natural that ONR should approach NBS with such a proposal, since ONR had been temporarily supporting the famous NBS, Mathematical Tables Project, originally a WPA project but financed during the war on OSRD funds. Also, in the spring of 1946, the Census Bureau had requested NBS to construct a large automatic digital machine suitable for the preparation of census reports. Therefore the nucleus for the sort of activity envisioned in Comdr. Wakelin's report was already present at NBS early in 1946.

A year of cooperative study ensued, including various conferences with possible clients and applied mathematical groups all over the country. The Air Materiel Command became interested, and made certain commitments in connection with its machine development program. This support, along with less substantial expressions of interest from other agencies, fulfilled the ONR requirements for outside participation. In the summer of 1947, NAML was established as a new division of NBS. It was organized into four branches: the Institute for Numerical Analysis in Los Angeles; the Computation Laboratory in Washington (to carry on the work of the Mathematical Tables Project but with more emphasis on problem-solving and less on tables); the Statistical Engineering Laboratory in Washington; and the Machine Development Laboratory in Washington.

The Institute was conceived as "the focal point in the organization for basic research and training in the types of mathematics which are pertinent to the efficient exploitation and further development of high-speed automatic digital computing equipment. A secondary function is to provide a computing service for Southern California and to give assistance in the formulation and analytical solution of problems in applied mathematics." The functions of the Institute are identified in more detail in a Prospectus for the NAML issued early in 1947, which was the basis of reference (2). They were specified there as follows:

"a) Plans and conducts a program of research in pure and applied mathematics aimed primarily at developing methods of analysis which will permit the most efficient and general use of high-speed automatic digital computing machinery.

"b) Conducts training programs for personnel of industry, Government agencies, and educational institutions, in the theory and disciplines needed for the full exploitation of high-speed automatic digital computing equipment.

"c) Studies and formulates requirements for the intelligence and internal organization which high-speed automatic computing machinery should have; develops overall performance specifications for such machinery.

"d) Serves as a center at which competent scholars can explore the usefulness of high-speed automatic digital computing machinery in their own fields of interest.

"e) Formulates requirements for further mathematical tables and other aids to computation; reviews the overall program of NAML with regard to the production of such objects and advises the Administration and Executive Council[4] accordingly.

"f) Reviews, analyzes, and, as necessary, assists in the mathematical formulation of problems in applied mathematics of the more complex and novel type arising in outside laboratories.

"g) Provides a computing service containing both standard equipment and high-speed automatic equipment (when available) for local industries, educational institutions, and Government agencies.

"h) Assists, and conducts liaison with, related programs in local educational institutions.

"i) Maintains a consulting service on special problems in applied mathematics.

"j) Prepares reports and monographs giving the results of the research described above; also prepares training manuals, bibliographies, and indices."

The Institute is divided into the Research section and the Mathematical Services section. The Research Section has from the beginning been supported almost exclusively by ONR, but now has some research for the Air Comptroller, USAF, underway. Also, a small subsidy from NBS pays for a part of the necessary experimental computing.

The Mathematical Services Section operates the computing service mentioned in the quotation above, and has recently completed the construction of a large-scale automatic digital computing machine under the direction of Dr. Harry D. Huskey. The machine, which is called the NBS Western Automatic Computer (SWAC for short), operates in the parallel mode and uses an electrostatic memory system, employing the Williams principle. The electrostatic memory was started with a 256-word capacity, soon to be increased to 512 words. A rotating-drum magnetic memory of 10,000 word capacity is also being added.

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[4]Now reconstituted as the NBS Applied Mathematics Advisory Council. The Council is referred to again below.

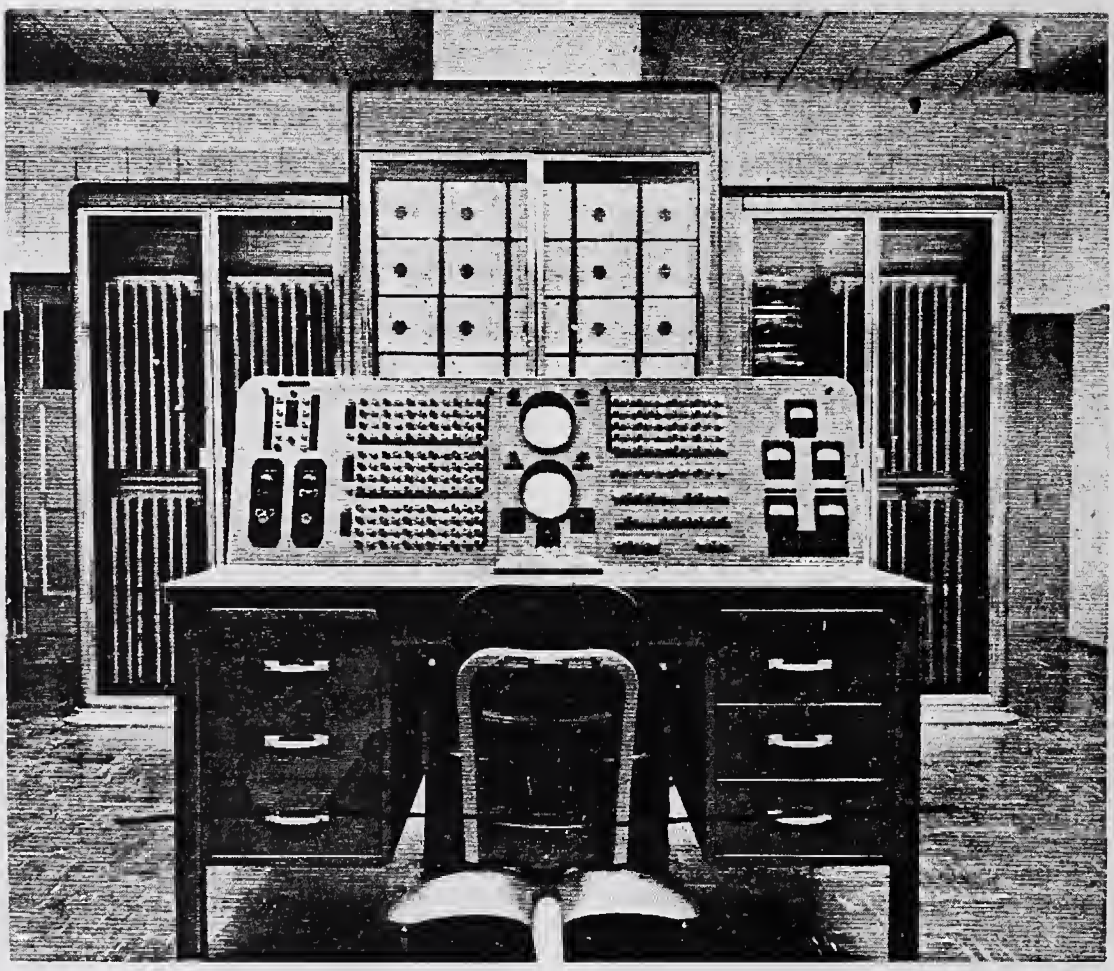

Fig. 1. The NBS western automatic computer (SWAC)

This machine was financed entirely by the Office of Air Research (OAR), Air Materiel Command, following its decisive role in the planning stages of NAML.

No computing service can be operated effectively from reimbursements obtained after (and in the Government, this sometimes means long after) problems are solved. Some form of advance financing is quite essential, both to pay for backlog work and to provide working capital. This critical form of budgetary support has been provided by OAR to the Institute's computation laboratory ever since the beginning. In addition, when OAR made its original decision to support construction of the SWAC, it provided that a certain minimum amount of time should always be available to the ONR research program.

When taken with certain aspects in the activities of the Eastern branches of the NAML, the interagency participation in the NAML program forms an ideal example of cooperation between Departments of the Executive Branch of the Government, and as such, has received favorable comment from the Bureau of the Budget. An important element in this picture is the NBS Applied Mathematics Advisory Council, which reviews and guides the program.

To return now to the Institute: Computers like to say there has been nothing new discovered in numerical analysis since Gauss. There have been many advances in other fields of mathematics, pure and applied, since that time. Can it be because professional mathematicians were not interested in numerical analysis and the field was left to amateur mathematicians to develop? Whatever the answer, it has been considered essential to adopt a very fundamental approach in setting up the program of the Institute. In particular, a policy of staffing the Research Section mainly with competent professional mathematicians has been rigorously followed. It has been found desirable, however, to have at least one theoretical physicist and one expert in classical applied mathematics on the staff, so that advice can readily be obtained as to profitable directions in which to work.

Every care has been taken to provide attractive working conditions. For example, most of the senior members of the research staff have private offices; experienced typists are on hand for typing mathematical manuscripts; desk computing machines are readily available for those who need them; and a conscious effort is made to insulate the scientists from administrative red tape.

The Institute staff members have faculty privileges in the excellent UCLA library nearby; but a book in hand is worth two on somebody's desk in a fraternity house six blocks away, and it was soon clear that a good working collection should be built up in the Institute. The aim has been to make it pre-eminent in all items bearing directly on numerical analysis, both ancient and modern, and to make it outstanding in up-to-date material in mathematical analysis and applied mathematics. Secondary emphasis has been placed on theoretical physics and certain branches of pure mathematics, such as abstract algebra, although outstanding new books in such fields are ordered. The Institute Library has been fortunate in picking up back editions of the standard mathematical journals; for example, it has about all volumes of Mathematische Annalen back to vol. 1 in 1869.

The computing equipment of the Institute, in addition to the SWAC, consists of hand machines and punched-card machinery. The punched card installation includes an IBM Card Programmed Calculator (this is an electronic computing machine with a substantial internal memory), a 604 Calculating Punch, two 602A multipliers, a tabulator, and various supporting items. The research mathematicians have easy access to the machines at all times (there is no restricted area), and unless an emergency arises they usually get prompt action on any experimental computing.

The significance of the Institute as a training center ranks high, and care has been taken to implement this aspect of the plans. Short-term appointments of senior research workers have introduced scholars in various fields to the facilities of the Institute (see Item (d) in the list of functions). There have also been many visitors to the Institute whose travel and expenses were paid for by their own institutions. Scientists are always welcome, and the visitors' book looks like a "Who's Who" of the mathematical world. Like the Institute for Advanced Study in the East, the Institute for Numerical Analysis in the West is sure to be visited by mathematical travelers in this country.

More formal educational programs are conducted, too. The Institute collaborates with UCLA on various courses. A particularly ambitious program of graduate mathematical instruction in modern numerical analysis is being given cooperatively by the Institute and UCLA in the summer of 1951. Fellowships of two types are also offered: summer studentships aimed at familiarizing graduate students of mathematics and physics with the Institute, and thesis fellowships lasting one or more years, to enable a graduate student to complete a Ph.D thesis. A number of symposia, colloquia, and short, intensive courses on modern automatic computing machinery have also been organized from time to time.

Credit for recognizing the need for fundamental research by the Institute should go particularly to Dr. Mina Rees, Director of the Mathematical Sciences Division of ONR. She saw that the great activity in developing automatic digital computing machines in this country was not being adequately paralleled by theoretical investigation aimed at finding out how best to use them. Indeed, it seemed that many of them would probably stand idle altogether too much of the time while mathematicians catch up with them.

The ONR research program is, and always has been, the heart of the Institute. The type of problem described at the beginning has had a great deal of attention in the past, but further studies are necessary to evaluate the older results in their relation to modern automatic machinery. Moreover, there are many significant types of problems, for which it is doubtful if appropriate methods exist at all, so that really difficult cases may demand completely novel approaches.

A type of problem which has received much attention at the Institute is that of finding the lowest eigenvalue of the time-independent Schrödinger equation

$$\frac{1}{2}\Delta u - Vu + \lambda u = 0, \tag{1}$$

where V and the unknown u are functions of several independent variables, Δ is the Laplacian operator, and λ is a constant. The boundary conditions are that u tends to zero as any coordinate approaches infinity, that the squares of the first partial derivatives of u are each integrable over the entire plane, and the first partial derivatives have simple discontinuities at the origin. The problem is important; in the theory of wave mechanics, the eigenvalues of this equation represent the energy states of a system whose potential energy function is V. Moreover, if suitable methods could be developed for this equation, they might be applicable to more complicated equations arising in the study of nuclear forces.

Discussion is restricted to one dimension, for simplicity of notation. A classical way of finding the lowest eigenvalue λ_1 of such an equation is the so-called Rayleigh-Ritz method, which is based on the theorem from the calculus of variation that

$$\lambda_1 = \min \frac{\int_{-\infty}^{+\infty}\left\{\frac{1}{2}(v'(x))^2 + V(x)(v(x))^2\right\}dx}{\int_{-\infty}^{+\infty}(v(x))^2\,dx},$$

where v(x) is any function belonging to a class of admissible functions. To apply this theorem calls for a grand search among all classes of admissible functions for the one that will minimize the expression on the right. The search can be systematized and abridged (the Rayleigh-Ritz method does that), but is still apt to be lengthy, particularly in many dimensions.

For the helium atom there are six independent variables x. The lowest eigenvalue was found with great accuracy by E. A. Hylleraas (3) in 1930, using the Rayleigh-Ritz method, after a celebrated and back-breaking computation. For more complicated atoms, this method seems to be practically out of the question on currently available computing machinery.

In another category of classical methods for solving such problems, the fundamental technique consists in replacing the differential equation by a corresponding one in finite differences and finding the eigenvalues of the latter. A good example of this attack will be found in (4). This essentially changes the problem to one of finding the eigenvalues of a matrix; but in may dimensions that in itself can be a most difficult task. Then too, very little is known about the relations between the eigenvalues of the difference equation and those of the differential equation. All in all this seems to be a problem for which it is worthwhile searching for new methods.

A fruitful hunting ground for such methods is in probability theory. It has been known for many years [for a brief history, see (5)] that the distribution functions associated with certain random walks provide exact solutions of certain difference equations, and asymptotic solutions of the related differential equations. One of the first applications of Monte-Carlo methods to finding eigenvalues was made at the Institute for Numerical Analysis by Professor Mark Kac during his tour of duty there in the summer of 1949. He made use of the following relation derived in his study of certain Wiener functionals:

$$\lambda_1 = \lim_{t \to \infty} \frac{\log E \left\{ \exp\left(-\int_0^t V(X(\tau))d\tau\right) \right\}}{t}, \qquad (2)$$

where E means expected (i.e. mean) value, and X(t) is a Wiener process with mean zero*. The expected value appearing in the formula is then evaluated by the following procedure: A random walk starting at the origin with a pre-assigned number n of elementary steps is executed and a score is tallied after each step according to the following rule: If S_k represents the position of the walk at the k-th step, the score $V(S_k/\sqrt{n})$ is tallied after the k-th step. After nt steps, the scores are all added and divided by n to obtain a final score. By another theorem of Dr. Kac (6), the distribution of the final scores approaches that of the integral in eq. 2. Therefore to evaluate approximately the right-hand side of eq. 2, it is only necessary to perform the random walk and get the final score many times, then raise e to the power of each final score, and average the results.

Clearly there are several sources of error in this approximation. There is the statistical error in the final average; there is the error in using the random walk approximation to the Wiener functional, and there is the approximation involved in using eq. 2. In the experiments of 1949, good results were obtained for the case $V(x) = x^2$ (the harmonic oscillator); these are reported in (7). The method has worked satisfactorily for the hydrogen atom, but the more difficult case of helium calls for further research into the nature of the approximations.

Another random walk method which avoids the theory of Wiener functionals was developed by Dr. W. Wasow (8) of the permanent staff of the Institute. It is so simple that up to a certain stage the errors of approximation can be studied analytically more easily than in the Kac method. However, the Wasow method belongs in a sense to the category of difference equation methods; it replaces (1) by the corresponding difference equation and approximates its eigenvalues, not those of eq. 1. Thus the method has the disadvantage shared by all difference equation methods and identified above, i.e., the dubiousness of the relation between the eigenvalues of a differential equation and those of the corresponding difference equation.

Dr. Wasow sets up the simplest possible symmetric random walk in the space of the independent variables, very much as Dr. Kac does, and proceeds to cumulate a multiplicative score as the walk goes on. If P and Q are two points of the lattice, and the walk starts at P, then he lets $g_n(P,Q)$ be the mean value of the score at Q if the walk arrives at Q in exactly n steps. Under certain conditions, the quantity $G(P,Q) = \sum_0^\infty g_n(P,Q)$ exists; this is analogous to the Green's function of the original differential equation. Obviously $g_n(P,Q)$, and, in fact, $g_n{}^*(P) = \sum_Q g_n(P,Q)$, can easily be estimated by performing the walk many times and keeping track of the score at each step. Wasow then sets up a generating function,

$$\psi^*(P, r) = \sum_0^\infty r^n g_n{}^*(P).$$

*i.e., X(t) for a single value of t is a random variable with the density function

$$\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

and for two or more values of t, say $t_1 \le t_2 < t_3$, the joint distribution of the corresponding random variables is such that $X(t_1)$ is statistically independent of $X(t_3) - X(t_2)$.

and shows that its poles have a very simple direct relation to the sought-for eigenvalues. It then remains only to estimate the poles, knowing the coefficients of ψ^*. In the case of a bounded fundamental region, this is facilitated by the fact that ψ^* is a rational function of r. Experiments on the Wasow method are now commencing.

Another interesting method, perhaps closer to Kac than to Wasow, was proposed by Dr. Robert Fortet (9) in a recent tour of duty at the Institute. He considers the analogous problem in integral equations (it is well known that (1) can be reduced to an integral equation if the Green's function of (1) is known). Dr. Fortet's method is based on an interesting theorem which was first proved by Kac and Siegert, and which he was the first to prove in all generality. It runs as follows: Consider the Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b G(x,\xi)u(\xi)d\xi,$$

Let $D(\lambda)$ be its Fredholm determinant. (The solutions of $D(\lambda)=0$ are the desired eigenvalues.) Let $X(t)$ be a real Gaussian process whose covariance function is G. Then

$$E\left[\exp(iv\int_a^b X^2(t)dt)\right] = \left[D(2iv)\right]^{-\frac{1}{2}}, \quad v \text{ real}.$$

All of these results depend on really sophisticated mathematical theory. It is not only the search for completely novel methods, however, that requires deep mathematical theory. For example, the Institute has devoted much thought to developing new methods of finding the eigenvalues of finite matrices. Although the methods studied at the Institute have generally fallen into the classical categories, nevertheless some serious mathematics has been required in proving the required convergence theorems. [See for example (10)].

In addition to solving large systems of linear equations, and calculating eigenvalues, a third and now principal area of Institute research is the development of techniques for numerically solving differential equations. The orientation came from certain difficult problems in non-linear parabolic differential equations brought into the Mathematical Services Section by nearby Naval scientific laboratories, notably the Naval Ordnance Test Station, China Lake and Pasadena. Monte-Carlo methods have been experimented with at some length in certain simpler cases, notably in the Dirichlet problem and conformal mapping. [A general exposition of the available techniques is given in (5)].

Problems worked on so far by the Institute are described periodically in quarterly reports to ONR and in the widely available public quarterly report of the NAML, entitled "Projects and Publications of the National Applied Mathematics Laboratories." (Copies will be sent regularly on request to any Naval activity having a need for the information.)

Detailed results have been prepared as papers, of which more than 100 have been submitted to the standard journals since 1 January 1948 for publication. A bibliography of the publications of the Institute to date will be found at the end of this article.

The Institute is an example of a successful venture of the Government into <u>fundamental</u> research. What is its usefulness in a time of more or less complete mobilization for defense? The question was foreseen during the planning days of the NAML and was answered in the Prospectus for the organization, referred to earlier. Of the four main guidelines for the program of the proposed center, the last was identified as follows:

"It should undertake to maintain a reservoir of personnel trained in applied mathematics which can be drawn on in case of a national emergency, and should at the same time develop disciplines and tools to facilitate the conversion of the nation's peace-time scientific manpower to emergency uses."

This, of course, applies as much to the Institute for Numerical Analysis as to the larger organization. The permanent staff, the substantial body of alumni, and the general good-will among scientists which the Institute has built up, will all serve as an important asset of scientific manpower if and when needed. Special problems of immediate importance may then be undertaken in addition to comprehensive, fundamental attacks on problem types, and some of the work will, of course, become classified. The familiarity which the research personnel and the visitors have obtained with computing machinery in their experimental work will be put to good use. But the economical and advantageous solution in mathematics is the one which applies to a whole class of problems and not just to the special one in hand. It is hoped that, however pressing the demands are on the Institute for immediate results of the "quick and dirty" kind, the organization will always continue to find time to do work of broad usefulness in the science of modern high-speed computing.

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4. L. Ahlfors. Numerical methods in conformal mapping.
5. Conformal representation of simply- and multiply-connected regions, by L. Kantorovitch and others; translation from the Russian by W. Seidel.

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APPENDIX C

THE SWAC

The building of the SWAC was one of the important accomplishments of INA. So far in the text we emphasized the mathematical accomplishments of INA and said very little about the construction of the SWAC. To complete the story of INA we now include an article by Harry D. Huskey describing the story of the SWAC. To obtain a comprehensive overview of computing machines developed at about this time and earlier, we refer the reader to the important paper by Mina Rees entitled "The Federal Computing Machine Program," *Science* **112** (1950), 731–736.

The SWAC: The National Bureau of Standards Western Automatic Computer

HARRY D. HUSKEY

1. Background

The SWAC had its beginning at the 19 October 1948 meeting of the Applied Mathematics Executive Council held at the National Bureau of Standards (NBS) in Washington, D.C. This Executive Council served as an advisory body to the National Applied Mathematics Laboratories, which was a division of the NBS.

The Mathematics Laboratories had been established in 1945 through a "suggestion" by the Navy Department to the Director of the Bureau of Standards, Dr. Edward U. Condon. The Navy hoped that the Bureau would establish a centralized national computation facility, equipped with high-speed automatic machinery, to provide computing service to other government agencies and to play an active part in the further development of computing machinery. Dr. Condon complied, setting up the National Applied Mathematics Laboratories, with Dr. John Curtiss as Chief. The Laboratories were to have four main parts: the Computation Laboratory, the Machine Development Laboratory, and the Statistical Engineering Laboratory, all in Washington, D.C., and the Institute for Numerical Analysis (INA), a field station to be located near some university in California.

The success of the ENIAC had excited mathematicians and other scientists to the possibilities now opening before them. No company was yet turn-

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ing out electronic computers, but several had become interested in trying. University scientists, encouraged by the University of Pennsylvania's success in the field, were also attempting to build computers for their own use. Government agencies, quick to see the potentials of an electronic computer, were eager to acquire one. However, the field was new, there was no background of experience, and no one was absolutely certain what type of computer would best suit his purpose, or even what company was most likely to build a workable computer within a reasonable time. Therefore, government agencies were glad to ask the NBS to assist them in negotiating with computer companies. In early 1948, the Bureau had begun negotiating with the Eckert–Mauchly Computer Corporation and the Raytheon Corporation, and later with Engineering Research Associates.

The computers were slow in being developed. New techniques were being tried and often they did not work as well, or as soon, as had been first thought, or hoped. The personnel of the Applied Mathematics Laboratories became impatient with this slow development, and decided that they could build one faster with the help of the Electronics Laboratory at the Bureau. Also, it had become clear that in order to be able to judge effectively the probability of a new technique working they would need more "hands-on" expertise. Dr. Edward Cannon and the author convinced Dr. Curtiss that this "gamble" was worth trying, and Dr. Mina Rees of the Office of Naval Research backed them up. This was in spite of the advice of a committee, consisting of Dr. George Stibitz, Dr. John von Neumann, and Dr. Howard Aiken, which had been asked by Dr. Curtiss to consider the Bureau's role in the computer field. Their advice had been that the NBS shouldn't really work on computers, but should confine its work to improving components.

In May 1948, the decision was made at the Executive Council to build a machine for the Bureau's own use in Washington. At that meeting it had also been decided that the Bureau should buy three UNIVAC's which were being developed by the Eckert–Mauchly Computer Corporation. One of these was to go to the Census Bureau, one to the Air Materiel Command in Dayton, Ohio, and the third to the INA. Later, due to a security problem that had arisen in the company, it was decided that the military funds could not be used to purchase UNIVACs. This reopened the question of procuring a computer for the Air Materiel Command and for the INA. Thus, at the October 1948 meeting of the Executive Council it was decided that the Bureau should build a second computer at the Institute for Numerical Analysis, which had by now been located in a reconverted temporary building on the campus of the University of California at Los Angeles. This machine was to be built under the direction of the author, who had joined Curtiss's group in January 1948. He had spend the previous year at the National Physical Laboratory in Teddington, England, working under Alan Turing with James Wilkinson and others on the Automatic Computing Engine (ACE) project. He had been offered the job there on the recommendation of Professor Douglas Hartree, whom he had met while working on the ENIAC project.

2. The Institute for Numerical Analysis

In December 1948, the author transferred to the Institute for Numerical Analysis, and in January 1949, work started on the INA computer. The computer at the NBS in Washington followed the EDVAC (University of Pennsylvania) design using mercury delay lines for memory. The Executive Council, which included representatives of the U.S. Air Force and the Office of Naval Research, felt that the NBS should not build the same type of machine that others were already building, or planning to build. The author had become interested in the possibility of using cathode-ray tubes for storage while in England, where he had seen the work being carried on at Manchester University under Professor F. C. Williams, and the proposal was made that this type of computer be built at INA. As a precaution, the Council wanted it to be designed in such a way that if it didn't work it could be converted to a magnetic drum computer. Of course, we had no doubt that it would work.

Finances were tight. Three hundred thirty thousand dollars had been transferred to the NBS to cover the cost of both the INA computer and the one for the Air Materiel Command (which was to be contracted to the Raytheon Corporation). Dr. Curtiss noted in his 1953 progress report on the Mathematics Laboratories, "The project was handicapped throughout by having much too tight an annual budget"[1].

At INA an empty room was given to the author and he was told to "go ahead." Not only did personnel have to be recruited to assist in designing and constructing the computer, but also machine shop equipment and supplies had to be procured from scratch. Fortunately, the Bureau had just completed a study contract with Eckert–Mauchly so we asked for all the machine tools acquired on that contract to be shipped to Los Angeles. There was also the race against time. After all, part of the justification for the NBS's building its own computers was the slowness of the would-be commerical suppliers in putting workable machines on the market. Consequently, construction of the computer began even while the study of the general machine organization and logical system was still under way. The development of the machine system, circuitry, and building techniques proceeded simultaneously with the actual construction of the machine.

The assembly of the computer was completed by July 1950. It was formally dedicated on 17–19 August. The opening session on the afternoon of the 17th featured speeches by Dr. Condon; Colonel F. S. Seiler, Chief of the Office of Air Research, USAF; Dr. L. N. Ridenour, Dean of the Graduate

School at the University of Illinois; Dr. Curtiss; and the author. The real highlight of the program, of course, was a demonstration of the computer.

The second day consisted of a symposium, opened by Dr. Condon, on the applications of digital computing machinery to scientific problems. Paul Armer, Leland Cunningham, Samuel Herrick, Stanley Frankel, Derrick Lehmer, and Jerzy Neyman were among the speakers. The third day was spent mainly in demonstrations of the SWAC.

3. Naming the Computer

The name of the computer had undergone several changes during its construction. In a talk that the author gave at the Second Symposium on Large-Scale Digital Calculating Machinery, held at the Computation Laboratory at Harvard University on 13–16 September 1949, the machine was called the ZEPHYR. This name had been chosen to emphasize the modest nature of the effort, "a gentle wind from the west," in contrast to other projects carrying names such as TYPHOON, HURRICANE, WHIRLWIND. Good-natured rivalry with the group at NBS in Washington caused them to suggest SIROCCO (a hot wind from the desert) as a substitute for ZEPHYR.*

In retrospect it was clear that ZEPHYR was not a strategic choice of name. Hence, the very prosaic name, Institute for Numerical Analysis Computer, was used for a time, and appears in an article published in [3]. Early in 1950, someone in the administration at NBS suggested that the names of the computers being built at the Laboratories in Washington and at INA be tied together to the glory of NBS. Hence came the name National Bureau of Standards Western Automatic Computer, which in the style of the times lent itself nicely to being shortened to the name SWAC. Similarly, the NBS computer in Washington was initially called the NBS Interim Computer. Later it became the National Bureau of Standards Eastern Automatic Computer or SEAC. It was constructed under the direction of Dr. Samuel Alexander.

4. Project Organization and Staff

The project was divided into three major parts: memory, arithmetic, and control. B. Ambrosio, with the help of Harry Larson, was to handle the memory. Bill Gunning visited from Rand Corporation and helped with the memory until he broke a leg skiing. Edward Lacey worked on the arithmetic unit, making use of the type of circuitry developed at MIT on the WHIRLWIND project. David Rutland worked on the control unit, and we pro-

* Independently, H. Zamenek of Vienna, Austria, named the machine built under his direction MAILUEFTERL (gentle wind, also as a joking comparison).

ceeded to put together a computer. R. Thorensen joined the group later and worked on the magnetic drum. Other parts such as power supply and input–output, all being more straightforward (requiring less development), were handled by the group in a less explicit way

By July 1950, the staff consisted of three engineers, three junior engineers, and four technicians, in addition to the author. Everyone knew that what was being done was pioneer work on a new frontier and excitement abounded. A good-natured comradely rivalry developed between the two projects at the Bureau, as well as with similar projects elsewhere. There was a good deal of open exchange of information and techniques, and no one doubted that significant progress was being made on all fronts. The staff worked long and irregular hours uncomplainingly, spurred on by the eager interest of the distinguished internationally known scientists working at the INA. All of them were eagerly looking forward to having such a tool as the SWAC at their disposal.

The computer was to be parallel, using for its memory standard commerically available cathode-ray tubes (in contrast to some projects that were experimenting with specially built tubes). This decision was made both in the interests of time and of money. In fact, components that were mass produced commerically were used wherever possible in the SWAC. The reasoning behind this decision was that they could be easily replaced, were relatively economical, and, in general, could be expected to have more reliability. To further assist in servicing and to decrease the down time for maintenance, it was also decided to have all circuitry on removable plug-in chassis, with spare plug-in units for about 80% of the computer. Thus, in case of the failure of some component the faulty chassis could be removed and replaced by a spare one. This type of construction was especially important in those early days when no one knew just how much reliability one could expect from the components. Laboratory test equipment was constructed so that faulty chassis could be repaired in the laboratory without use of computer time. The majority of the plug-in chassis contained an average of 10 tubes each; however, the magnetic drum circuits, which were built later, were smaller, usually having a single tube per plug in unit.

Besides the 37 cathode-ray tubes used in the memory, the SWAC contained 2600 tubes and 3700 crystal diodes. The average tube life was between 8000 and 10,000 hours. Most tube failures resulted from low emission and intermittent shorts, and not from heater failure.

5. Memory

As noted earlier, the author had visited F. C. Williams and his staff at Manchester University, and observed there the method of storing information in cathode-ray tubes. One of the main appeals of the Williams system was the high speed of computation possible because of random access to memory locations. Numbers could be transferred in parallel instead of by serial pulse trains. However, the machine being designed at Manchester was serial by bit in operation, which meant that information was transferred to or from one tube at a time. In order to take full advantage of the possible speed of the Williams tube storage the SWAC was designed to operate in parallel. That is, information would be transferred in and out of all memory tubes simultaneously. This may have increased the headaches during construction but it did pay off in speed. At the time of its dedication in August 1950, the SWAC was the fastest computer in existence, being able to do 16,000 three-address additions or 2600 such multiplications per second.

The main disadvantage of the cathode-ray type of storage on the SWAC was its relatively small size. The SWAC stored 256 words of 37 binary digits (bits) each. To do this 37 cathode-ray tubes were used, with the various bits of a particular number being stored in corresponding positions of each of the tubes. At first it had been hoped that it would be possible to push the high-speed memory above 256 words (to 512 or even 1024 words). However, memory difficulties prevented this from happening. In this type of memory the individual digits of information were stored as spots of charge that existed over small areas inside the face of the tube. These spots were arranged in a rectangular array on the face of the tube. Two different charge distributions, providing the two states needed to represent a binary digit, could be produced at each spot. These spots had either a dot or a dash appearance, the dot corresponding to a 0 and the dash to a 1. A monitoring tube mounted on the console showed the dot–dash pattern in terms of zeros and ones on any one of the 37 memory tubes depending on an appropriate switch setting.

This type of memory required regeneration. Unless this was done the original charge pattern would tend to disappear over a period of time as the charged spots collected stray electrons. We always knew this would happen, so the memory cycle was made 16 μsec long, with an 8-μsec action cycle followed by an 8-μsec restore cycle. During the action cycle operands were transferred from the memory to the arithmetic unit, results were transferred back to the memory, or the next instruction was transferred to the control unit. In the restore, or regeneration, cycle one of the spots was restored to its initial value. Thus, in 256 restore cycles (about 4 msec) the whole memory would have been regenerated. Under these conditions, information could be stored in the memory indefinitely. A crystal-controlled oscillator regulated both the rate of regeneration of the memory and the synchronization of the control circuitry.

There were two main features of the Williams-tube type of memory that gave us considerable trouble, and prevented us from storing more information on each tube. One of these was the presence of so-called flaws on the inside of the face of the tube, which prevented it from storing information satisfactorily. We found that each particular tube had three or four such spots on it.

These spots turned out to be small carbonized particles of lint. The problem was that we had to build 37 tubes, deflecting all the 256 bits on the faces in unison so that no one of the bits would land on any of these carbonized spots. A storage spot might be alongside one of the flaws, and any slight drift might bring the spot onto it. It was only after it was too late to look for another supplier that we learned that our tubes were being manufactured in a reconverted mattress factory! That no doubt had increased our difficulties substantially; however, all commerically manufactured cathode-ray tubes had some flaws in them. To minimize this difficulty tubes were carefully selected, and the location of the memory array was adjusted so as to avoid bad spots. We also spent a lot of effort producing extremely stable power supplies, so that we could control drift for reasonable periods of time in order to avoid trouble from such spots.

Another major problem with the memory was "spill-over," or redistribution of charge. Each memory access generated a cloud of secondary electrons that "rained down" on the neighboring charged spots. This limited the number of times the neighboring spots could be read before the spot in question had to be regenerated. The term read-around ratio indicated how many times one could look at a given spot before the neighbors were ruined. A program might access spots in a given area of the tube many times before the regeneration occurred. One was between the devil and the deep blue sea—if one adjusted parameters in one direction flaws were less troublesome, but the read-around ratio would collapse. If one tried to improve the read around ratio with sharper focus, then flaws were more of a problem. It was spill-over that kept us from increasing the memory size above 256 words.

Since it was not practical to enlarge the size of the high-speed memory, it became imperative to have auxiliary storage. It was decided to use a magnetic drum for this purpose. At first it was planned to use a magnetic drum built by Professor Paul Morton at the University of California, Berkeley. In those early days drums were made from extruded aluminum tubing. There are stresses in the metal and, as time passes, the material flows. This happened to our drum. Although it was round when delivered to us, it was no longer round by the time we had built circuitry and tried to connect it to the computer. There was much folklore around, such as "haul the tubing around in the trunk of your car for six months so that it would be relaxed." Ultimately we purchased another drum from a commerical supplier and connected it to the computer. The drum used had 4096 words of memory. Average access time per word was about 500 μsec, in contrast to the cathode-ray tube access time of 16 μsec. Transfers to and from the drum were most efficiently handled in blocks of 32 words, which was the number of words on any track on the drum. The transfer would begin with the first number that became available and persisted for precisely one revolution of the drum. The drum rotated at 3600 revolutions per minute, so 32 words were transferred in approximately 17,000 μsec. Since there was no wait time in this mode, the access time per word was about 500 μsec. The drum transfer instruction also allowed for transfers of 8 or 16 words. However, these took the same time as a full 32-word transfer.

6. Arithmetic

The arithmetic unit used three 37-bit registers. A memory buffer register supplied operands from memory. There were 37 binary adders that could add this operand to the contents of the accumulator (the second register). The third register (called R) stored the multiplier and parts of the product during multiplication.

The maximum carry propagation time (through 37 bits) was 6 μsec. Negative numbers were stored in memory as sign and absolute value. Negative operands were complemented in the memory buffer. The timing was synchronous with sufficient time being allowed for maximum carry propagation. This timing very nicely matched the 8-μsec memory access cycle.

At the time of its dedication it was the fastest computer in existence, being able to do a three-adress ($C = A + B$) addition in 64 μsec and a similar multiplication in 384 μsec.

High-current low-impedance circuits were used in the arithmetic unit to give a fast carry so that multiplication would take the minimum time. Multiplication was by repeated addition with shifting occurring on an 8-μsec cycle. The least significant bits were processed first so that carry was at most across 37 bits. The use of high-current circuits caused substantial change in power supply current when operands were all ones versus all zeros. This change was a much as 15 amperes. A neat diagnostic test was discovered after some months of operation. It consisted of multiplying ones by ones for a tenth of a second, then running the diagnostic program the next tenth. Since the power supply filtering was designed only for three phase full wave (equivalent to 360 cycles) rectification, this change in load caused changes in voltage levels of about 20 volts. When the diagnostic routines ran under these conditions, no known failures occurred on general problems.

7. Input and Output

Initial input and output were by typewriter and punched paper tape. The typewriters (Flexowriters) and the tape stations required substantial maintenance and were terribly slow. Therefore, we soon connected an IBM collator (077) and a card punch (513) for input and output. The collator read cards at 240 per minute and the punch punched at 80 cards per minute. Seventy-eight bits (2 words) were read in from each row on the card and 11 rows were used. Substantial computing could be done between rows on the card.

8. Control

The SWAC used eight basic instructions: add: subtract: multiply rounded: product (two-word answers); compare; extract: input: and output. There was a variation of the compare instruction, which compared absolute values.

Two principles were followed in deciding upon this list of basic instructions. The first was that there should be as few instructions as practical in order to simplify the electronic circuitry of the computer, and to permit speedy construction. The second was that the instructions should be sufficiently general to do scientific computation. The SWAC used a four-address instruction. A floating-point interpretive system called SWACPEC was later developed, which made it easier for users to write programs.

9. Other Details

The SWAC was small in size compared to most of the computers of its time. The units were mounted in three connected cabinets, which were made to order by a local manufacturer. The memory and control were on one side and back-to-back with it was the arithmetic unit. The total size of these cabinets was approximately 12 ft wide, 5 ft deep, and 8 ft high. For esthetic reasons the cabinets had glass (shower) doors. The building was made of wood and had fire sprinklers mounted in the ceiling. This led to a good deal of kidding from the INA staff saying that there was something wrong in the relative location of the sprinklers and the shower doors.

The operating console was an ordinary desk with specially built panels mounted on its top surface. The actual operation of the SWAC took place from it. In addition to the memory monitoring tube already mentioned, there were neon lights and another cathode-ray tube that indicated the address in memory involving memory accesses during action cycles. The desk and cabinets were a tan color. The entire computer was located in a room 40 by 30 ft, with power supplies, a motor generator, and an air conditioner located elsewhere.

In the course of operation, certain logical facilities were added to make the operation more efficient. One of the more interesting of these was the addition of a loudspeaker and plug-in arrangement allowing the operator to "listen" to any of the instructions in a problem. For example, an alternate succession of add and subtract instructions produced an 8-kHz note. One of the problems run on the SWAC involved the generation of pseudorandom digits. The corresponding sequence of tones was christened the Random Symphony.

Being afraid of trouble from power line transients caused by other activities on the UCLA campus, from the beginning we expected to use a motor generator on the power line. The Engineering Department offered us a spare alternator. For some reason that we never quite understood (the

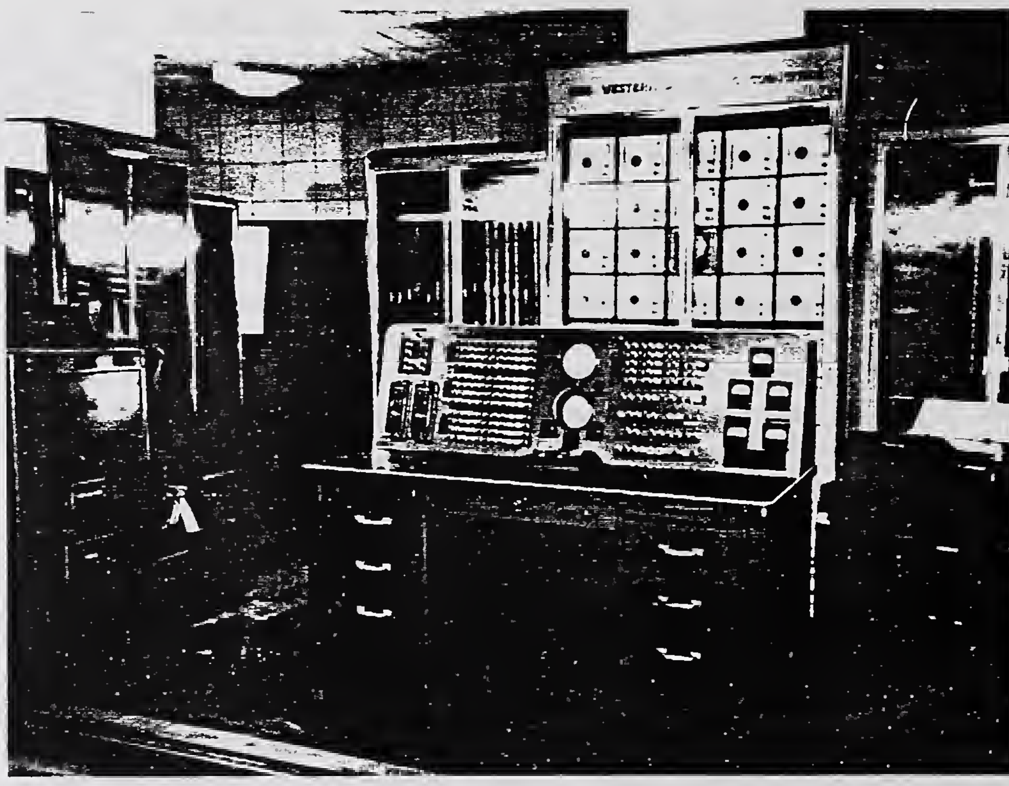

Figure 1

alternator was alleged to have the wrong number of poles) the output of the power supplies when using the alternator had more noise then when we connected directly to the power line. Ultimately, we replaced it with a larger regulated system with much more satisfactory results.

We had expected to cool the computer with the Los Angeles air. However, the germanium diodes turned out to be much more temperature sensitive then we had expected, so we added a cooling unit to the air system. By this time the total computer was very well integrated with the building. This only became a problem later when it was decided to move the computer to the UCLA Engineering Building.

10. Applications

By mid-1953, the SWAC was producing useful results 70% of the time that it was turned on, and was doing over 53 hours of useful computing per week. This was before the installation of the magnetic drum.

The SWAC was used in a research computing environment, and therefore the problems run on it tended to be quite large. Solution times as high as 453 hours were reported during its early operation.

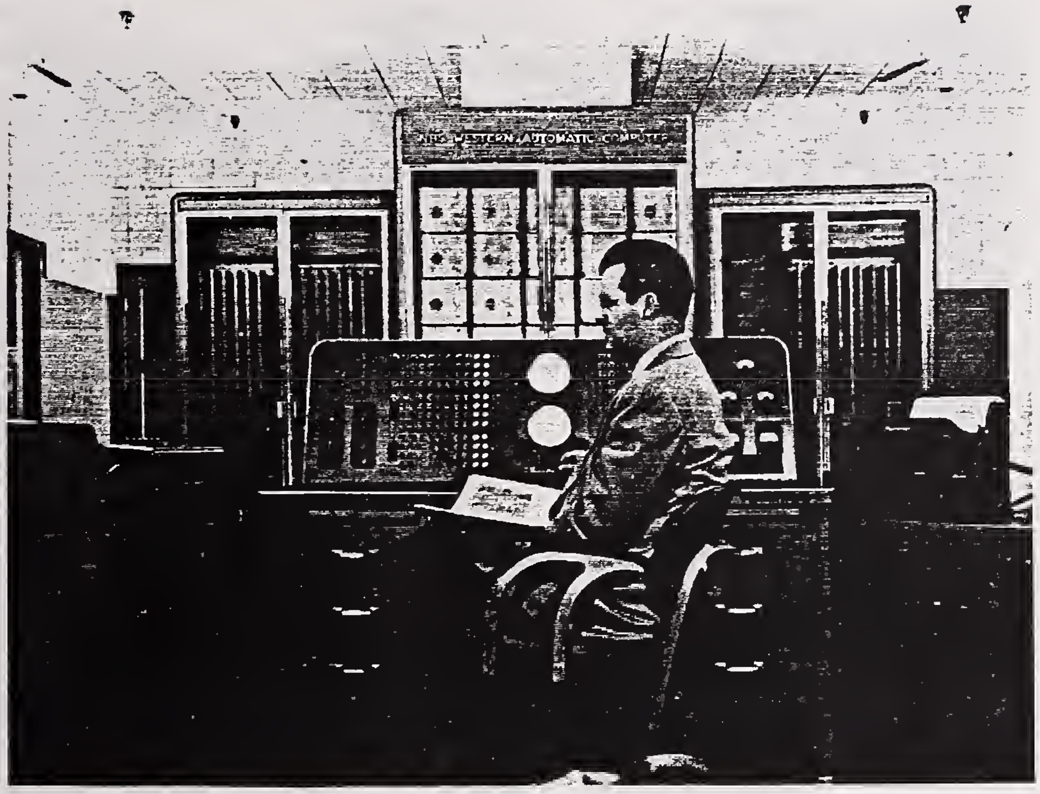

Figure 2

The INA was an exciting place as many scientists of international reputation spent a few days, weeks, months, and occasionally a year or more there. It was a crossroads for numerical analysts and early computer scientists. Some of the early problems included the search for Mersenne primes, the Fourier synthesis of x-ray diffraction patterns of crystals, the solution of systems of linear equations, and problems in differential equations. In addition to problems originating from the INA staff, the computer was also used to do problems for other government agencies.

One of the exciting problems in pure mathematics on which the SWAC worked was the study of Mersenne numbers, that is, numbers of the form 2^p-1, where p is a prime. These numbers, when prime, are related to the "perfect numbers" of the Greeks, numbers that are the sum of all their integral divisors excluding themselves. The list of values of p that yielded prime numbers up to that time was $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89$, and 107. Everyone was greatly excited when the SWAC added 521 to the list, and there was a real celebration when 607 was added about an hour later. In all, by June 1953, five values had been added to the list with the use of the SWAC as a result of systematic testing of all primes up to 2297.

Applied problems, such as the study of large-scale circulation patterns in the earth's atmosphere, were also run on the SWAC. In this problem 750,000

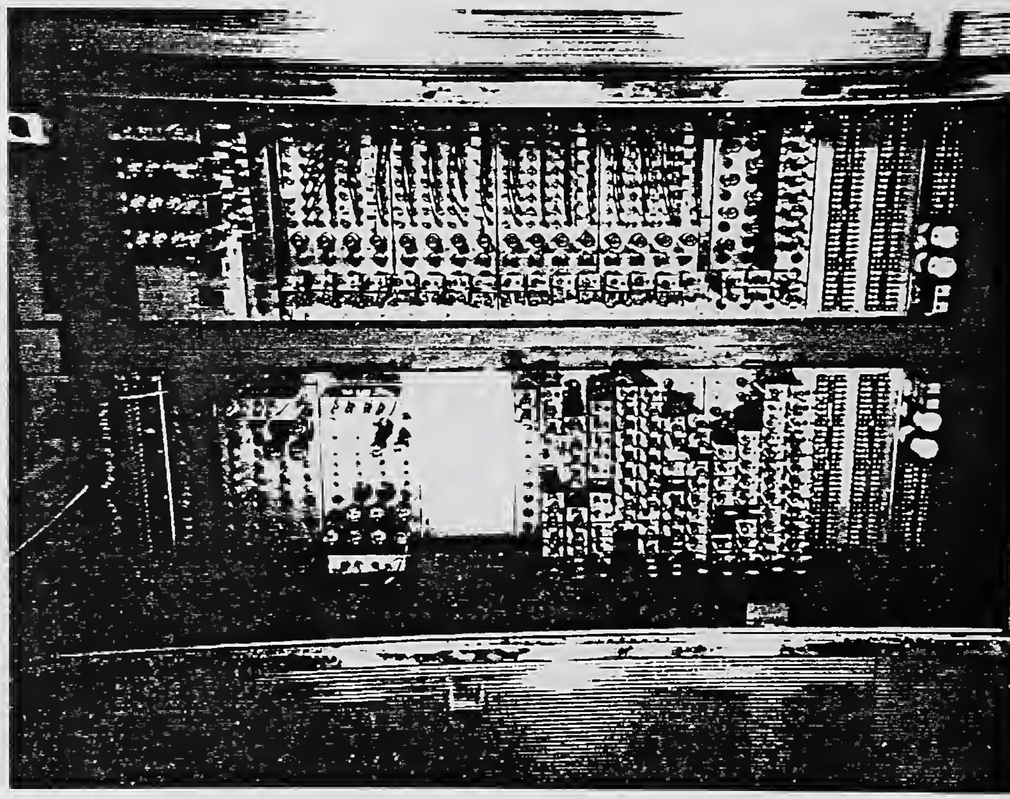

Figure 3

pieces of data were processed to yield a similar number of answers. SWAC spent 325 hours on this problem.

When the NBS ceased to support the INA in 1954, the SWAC was transferred to the University of California at Los Angeles and moved to the Engineering Building. There it continued in useful operation until it was retired in December 1967, at the age of 17. Parts of the SWAC are on exhibit in the Museum of Science and Industry in Los Angeles.

ACKNOWLEDGMENT

The author wishes to express his appreciation to Velma R. Huskey for assisting in the preparation of this paper.

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DIVISION OF NATURAL SCIENCES
UNIVERSITY OF CALIFORNIA
SANTA CRUZ, CALIFORNIA

APPENDIX D

SYMPOSIA AND TABLES

In this appendix we give the title pages and tables of contents of the volumes of the NBS Applied Mathematics Series which were written as a result of the activities at NBS-INA. A number of these are proceedings of various symposia held at INA, thereby giving a description of the program of these symposia. Others are outgrowths of these symposia. In addition, three of them describe important tables which were compiled at INA.

PROCEEDINGS OF
SYMPOSIA IN APPLIED MATHEMATICS

VOLUME VI

NUMERICAL ANALYSIS

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK TORONTO LONDON

1956

FOR THE AMERICAN MATHEMATICAL SOCIETY

80 WATERMAN STREET, PROVIDENCE, RHODE ISLAND

PROCEEDINGS OF THE
SIXTH SYMPOSIUM IN APPLIED MATHEMATICS
OF THE AMERICAN MATHEMATICAL SOCIETY

Held at the Santa Monica City College
August 26–28, 1953

COSPONSORED BY
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John H. Curtiss
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Library of Congress Catalog Card Number 50-1183

CONTENTS

UNITED STATES DEPARTMENT OF COMMERCE • Charles Sawyer, *Secretary*
NATIONAL BUREAU OF STANDARDS • E. U. Condon, *Director*

Table of Arctangents of Rational Numbers

John Todd

National Bureau of Standards
Applied Mathematics Series • 11

Issued March 30, 1951

(Reprinted January 4, 1965)

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1951

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WASHINGTON, D.C., 20402 - PRICE 70 CENTS (PAPER COVER)

Contents

NOTATION

By Arctan x and Arccot x, where x is a finite real number, we invariably understand the principal values of the many-valued quantities arctan x and arccot x, i. e., the values which satisfy

$$-\tfrac{1}{2}\pi < \text{Arctan } x < \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi < \text{Arccot } x < \tfrac{1}{2}\pi.$$

In the tables, for integral values of x, we write

$$(x) = \text{Arctan } x.$$

UNITED STATES DEPARTMENT OF COMMERCE • Charles Sawyer, *Secretary*
NATIONAL BUREAU OF STANDARDS • E. U. Condon, *Director*

Monte Carlo Method

Proceedings of a symposium held June 29, 30, and July 1, 1949, in Los Angeles, California, under the sponsorship of the RAND Corporation, and the National Bureau of Standards, with the cooperation of the Oak Ridge National Laboratory

National Bureau of Standards
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Problems for the Numerical Analysis of the Future

National Bureau of Standards
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PRICE 20 CENTS

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NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Construction and Applications of Conformal Maps

Proceedings of a Symposium

Held on June 22–25, 1949, at the
Institute for Numerical Analysis
of the
National Bureau of Standards
at the
University of California, Los Angeles

Edited by E. F. Beckenbach

National Bureau of Standards
Applied Mathematics Series • 18

Issued December 26, 1952

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Contents

UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Tables for Rocket and Comet Orbits

Samuel Herrick

National Bureau of Standards
Applied Mathematics Series • 20
Issued March 9, 1953

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1953
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Contents

UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Simultaneous Linear Equations and the Determination of Eigenvalues

Edited by L. J. Paige and Olga Taussky

Proceedings of a symposium held August 23–25, 1951, in Los Angeles, California, under the sponsorship of the National Bureau of Standards, in cooperation with the Office of Naval Research

National Bureau of Standards
Applied Mathematics Series • 29
Issued August 31, 1953

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. - - - - - - Price $1.50

Contents

UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues

Edited by Olga Taussky

National Bureau of Standards
Applied Mathematics Series • 39

Issued September 30, 1954

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. - - - - - - - - Price $2.00

Contents

UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Experiments in the Computation of Conformal Maps

Edited by John Todd

National Bureau of Standards
Applied Mathematics Series • 42

Issued January 20, 1955

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. - - - - - Price 40 cents

Contents

UNITED STATES DEPARTMENT OF COMMERCE • Sinclair Weeks, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Further Contributions to the Solution of Simultaneous Linear Equations and the Determination of Eigenvalues

National Bureau of Standards
Applied Mathematics Series • 49

Issued January 15, 1958

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. - - - - - - - Price 50 cents

Contents

UNITED STATES DEPARTMENT OF COMMERCE • Lewis L. Strauss, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Tables of the Bivariate Normal Distribution Function and Related Functions

Compiled and Edited
by the
National Bureau of Standards

National Bureau of Standards
Applied Mathematics Series • 50

Issued June 15, 1959

For sale by the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. - - - - - - - Price $3.25

Contents

EDUCATIONAL ACTIVITIES

From the beginning there was a broad educational program at INA. We have seen already, in Appendix D and in the text, that a series of symposia was sponsored by INA jointly with UCLA and/or other institutions. Members of INA were encouraged to participate in seminars and colloquia sponsored by UCLA and neighboring institutions. A large number of informal lectures and seminars were conducted by visitors of INA as well as by regular members of INA. These lectures and seminars normally dealt with topics related in a broad sense to numerical analysis and machine computations. Students and faculty of UCLA and members of other institutions were welcome to attend these meetings. As stated in the text, INA also sponsored a series of colloquia on a broad set of topics of interest to the group. The speakers could choose their own topics. Most of the speakers were visitors to INA. We have compiled the following chronological list of visiting colloquium speakers:

1948–49

A. S. Besicovitch	C. Brown
C. F. Davis	H. H. Germond
J. J. Gilvary	C. Hastings
Casimir Kuratowski	Sir Harold Spencer Jones

1949–50

E. T. Benedict	M. Fekete
R. P. Feynman	R. A. Fisher
O. Helmer	J. O. Hirschfelder
R. Isaacs	I. Kaplansky
S. Lefschetz	E. Penney
W. Prager	G. Szegö
A. W. Tucker	

1950–51

N. Aronszajn	A. Erdélyi
E. Gerjuoy	K. Knopp
G. Pólya	S. Sherman
J. J. Stoker	D. van Dantzig
I. J. Schoenberg	J. H. Curtiss
L. Bers	J. L. Walsh
J. B. Rosser	G. Fichera
E. Stiefel	F. J. Murray
J. G. van der Corput	L. M. Blumenthal
Y. L. Luke	S. P. Frankel
I. J. Schoenberg	W. T. Reid
R. Isaacs	A. J. Hoffman

1952–53

A. M. Ostrowski	A. T. Lonseth
G. A. Hunt	M. Kac
D. Ray	H. Levine
J. G. van der Corput	G. Pólya
E. W. Barankin	P. Erdös
A. Dvoretsky	J. Wolfowitz
M. M. Shiffer	N. Minorsky
R. de Vogelaere	S. Bochner

G. Pall	S. Lefschetz
J. Marschak	R. Radner
A. Baños, Jr.	F. A. Foster

INA also sponsored regular courses and seminars for its members. They were conducted by INA staff members for no additional compensation. They fall into two classes:

1) Research activities and within-training courses for the INA Staff to which UCLA staff members and other interested persons were usually invited. These were part of the regular workday of the instructors.

2) Formal UCLA courses in numerical analysis and closely related subjects, attended by INA staff members and UCLA students. The work days of the instructors of these courses were adjusted so that teaching them was in addition to the regularly assigned duties.

The following INA staff members and visitors acted as instructors of these courses at various times.

F. S. Acton	G. Blanch
S. S. Cairns	J. H. Curtiss
P. Erdos	W. Feller
R. P. Feynman	G. E. Forsythe
D. R. Hartree	H. D. Huskey
F. John	C. Lanczos
R. S. Lipkis	H. Luxenberg
W. E. Milne	T. S. Motzkin
A. Ostrowski	H. A. Rademacher
J. B. Rosser	D. S. Saxon
I. J. Schoenberg	T. Southard
R. Thorensen	C. B. Tompkins
J. G. van der Corput	W. R. Wasow
E. C. Yowell	

In addition to these courses, there were a second set of courses taught by INA staff members on their own time, not as a function of the INA, and ordinarily for additional compensation. These courses were sponsored by UCLA, UCLA Extension Division, and the University of Southern California. Some of the INA participants were:

B. F. Ambrosio	A. D. Hestenes
F. H. Hollander	M. Howard
H. D. Huskey	C. Lanczos
E. E. Osborne	D. S. Saxon
J. Schwinger	T. H. Southard
R. Thorensen	C. B. Tompkins
J. L. Walsh	W. R. Wasow
M. Weber	E. C. Yowell

Some members of INA held joint appointments at INA and UCLA. They usually gave a graduate course at UCLA in their field of specialization. In addition members of INA, such as, Forsythe, Motzkin, Tompkins, and Wasow participated in the Ph. D. theses programs at UCLA and USC.

The Educational Program at INA led to the writing of lecture notes in numerical analysis and related topics. These lecture notes were issued from time to time as working papers or as NBS Reports. Among these are:

G. E. Forsythe:	Numerical methods for elliptic partial differential equations
	Theory of selected methods of finite matrix inversion and decomposition.
J. G. van der Corput:	Asymptotic expansions

C. Lanczos:	Approximation by orthogonal polynomials
H. Rademacher:	Elliptic and modular functions
T. S. Motzkin:	Chebyshev polynomials and extremum problems
I. J. Schoenberg:	Partial differential and difference equations
J. Schwinger:	Quantum Dynamics
W. Wasow:	Introduction to the asymptotic theory of ordinary linear differential equations

Many of these were later developed into books. The preparation of these notes was often part of the duties of the graduate fellows.

Members of INA and NAR also participated in writing articles forming the "The Tree of Mathematics" edited by Glenn James of UCLA and published in 1957 in Mathematics Magazine. Their contributions were

E. F. Beckenbach:	Complex Variable Theory; Metric Differential Geometry
J. H. Curtiss:	Elements of a Theory of Probability
J. W. Green:	Theory of Functions of a Real Variable
M. R. Hestenes:	An Elementary Introduction to the Calculus of Variations
Olga and John Todd:	Applications of Systems of Equations, Matrices, and Determinants

Members of INA and NAR also participated in a series of lectures sponsored by the UCLA Extension Division. Their lectures were published in a book entitled "Modern Mathematics for the Engineer" edited by E. F. Beckenbach and published in 1956 by McGraw-Hill. The INA-NAR participants were J. W. Green, M. R. Hestenes, G. E. Forsythe, D. H. Lehmer, and C. B. Tompkins.

PERSONNEL

A large number of persons were officially connected with INA. We have compiled a list of these members in this appendix. We make no attempt to list the many visitors who came to INA as unofficial members. Among these were distinguished scientists, such as J. von Neumann, R. Courant, E. Teller, S. Lefschetz, Norbert Wiener, and many others. We had frequent visitors from abroad, particularly from England. Then, too, there were unofficial members of INA comprised of faculty and graduate students from UCLA and neighboring universities.

We now present the official list of personnel of INA.

Chief, AMD
(Located at NBS, Washington, DC)

John Hamilton Curtiss	1947–53
Franz L. Alt	1953–54

Assistant Chief, AMD
(Located at NBS, Washington, DC)

Edward W. Cannon	1947–54

Consultant to the Division
(Located at NBS, Washington, DC)

Olga Taussky-Todd	1949–54

She was an initial member of INA 1947–48.

Chief of the Computation Laboratory at NBS-Washington

John Todd	1949–54

Todd and his group collaborated with the staff at INA. Todd was an initial member of INA 1947–48.

Directors of INA

Douglas R. Hartree	Spring and Summer 1948
John H. Curtiss	1948–49
J. Barkley Rosser	1949–50
Fritz John	1950–51
Derrick H. Lehmer	1951–53
Charles B. Tompkins	1953–54

Assistant Director and UCLA Liaison Officer

Magnus R. Hestenes	1950–54

In this capacity he was ably assisted by William T. Puckett of UCLA. During this period, Hestenes was also the Chairman of the Department of Mathematics at UCLA.

Chief of Machine Development and Mathematical Services

Harry D. Huskey	1948–54

Assistant to the Director

Gertrude Blanch	1948–54	(Numerical Analysis)
Albert S. Cahn, Jr.	1948–53	(Administration)
Wilbur W. Bolton, Jr.	1948–49	(Administration)

Senior Researchers, Regular Members

The dates given are the initial dates of employment.

Forman S. Acton	1949	
Gertrude Blanch	1948	
George E. Forsythe	1948	
Magnus R. Hestenes	1949	(Part time)
Harry D. Huskey	1948	
Cornelius Lanczos	1949	
Theodore S. Motzkin	1950	
David S. Saxon	1950	(UCLA-Physicist)
Dan Teichroew	1952	
Charles B. Tompkins	1952	
Wolfgang R. Wasow	1949	

(Members of the Mathematical Services Unit also carried out research.)

Senior Researchers, Temporary Members

The dates given are the initial dates of employment. Some members served during several periods. Their permanent affiliations are enclosed in parentheses. Except where noted, they were Mathematicians.

Lars V. Ahlfors	1949	(Harvard)
Adrian A. Albert	1952	(Chicago)
Alfredo Baños, Jr.	1952	(UCLA-Physicist)
Edward W. Barankin	1952	(Berkeley)
Edwin F. Beckenbach	1948	(UCLA)
Leonard M. Blumenthal	1951	(Missouri)
Truman A. Botts	1952	(Virginia)
B. Vivian Bowden	1951	(England)
Alfred Brauer	1951	(North Carolina)
Richard H. Bruck	1953	(Wisconsin)
William E. Bull	1951	(UCLA-Linguist)
Kenneth A. Bush	1953	(Illinois)
Robert H. Cameron	1948	(Minnesota)
Randolph Church	1953	(Monterey)
Milton Dandrell	1950	(UCLA)
Robert P. Dilworth	1953	(Caltech)
Monroe D. Donsker	1949	(Cornell)
Aryeh Dvoretsky	1950	(Israel)
Paul S. Dwyer	1951	(Michigan)
Paul Erdös	1950	(Hungary)
Richard P. Feynman	1950	(Cornell)
Gaetano Fichera	1951	(Italy)
Donald E. Fogelquist	1951	(Sweden)
Robert Fortet	1950	(France)
William Feller	1949	(Cornell)
Jerry W. Gaddum	1951	(Missouri)
Herman H. Goldstine	1951	(IAS)
Richard A. Good	1953	(Maryland)

John W. Green	1950	(UCLA)
Robert E. Greenwood	1948	(Texas)
Dick Wick Hall	1952	(Maryland)
Marshall Hall, Jr.	1953	(Ohio State)
E. H. Hanson	1952	(Texas State)
G. A. Hedlund	1952	(Yale)
Douglas R. Hartree	1948	(Cambridge, England)
Samuel Herrick	1948	(UCLA-Astronomer)
Gilbert A. Hunt	1951	(Cornell)
Fritz John	1950	(New York)
Mark Kac	1949	(Cornell)
Irving Kaplansky	1952	(Chicago)
William Karush	1949	(Chicago)
Tom Kilburn	1951	(England)
Erwin Kleinfeld	1953	(Chicago, Ohio State)
Harold W. Kuhn	1953	(Bryn Mawr)
Derrick H. Lehmer	1951	(Berkeley)
Richard A. Leibler	1952	(Washington, DC)
Harold Levine	1952	(Harvard-Physicist)
Hans Lewy	1950	(Berkeley)
Arvid T. Lonseth	1952	(Oregon State)
Jacob Marchak	1953	(Chicago-Economics)
Edward J. McShane	1950	(Virginia)
William E. Milne	1948	(Oregon)
Frank J. Murray	1951	(Columbia)
Lowell J. Paige	1950	(UCLA)
Gordon Pall	1953	(Ill. Tech.)
Raymond P. Peterson, Jr.	1950	(UCLA)
William A. Pierce	1953	(Syracuse)
Alexander M. Ostrowski	1949	(Basel, Switzerland)
Victor A. Oswald, Jr.	1951	(UCLA-Linguist)
Hans A. Rademacher	1948	(Pennsylvania)
J. Barkley Rosser	1949	(Cornell)
Isaac J. Schoenberg	1951	(Pennsylvania)
Julian Schwinger	1952	(Harvard-Physicist)
Wladimir Seidel	1948	(Wayne)
Edward L. Stiefel	1951	(Switzerland)
Philip Stein	1951	(South Africa)
Jonathan D. Swift	1953	(UCLA)
Otto Szász	1948	(Cincinnati)
John Todd	1947	(London, England)
Olga Taussky-Todd	1947	(London, England)
J. G. van der Corput	1951	(Holland)
Joseph L. Walsh	1951	(Harvard)
James A. Ward	1952	(Utah)
Stephen E. Warschawski	1949	(Minnesota)
Alexander Weinstein	1951	(Maryland)
Charles Wexler	1952	(Arizona State)
Maurice V. Wilkes	1951	(England)
Jacob Wolfowitz	1952	(Cornell)

Junior Researchers and Graduate Fellows

The dates listed are the initial dates of employment.

John W. Addison, Jr.	1952
Charles E. Africa, Jr.	1951
Donald G. Aronson	1951

Fred Baskin	1952
Richard G. Cornell	1951
Richard E. Cutkosky	1950
Robert J. Diamond	1949
Robert C. Douhitt	1949
Harold P. Edmundson	1950
Ernest E. Elyash	1949
Stuart L. Fletcher	1950
Walter I. Futterman	1952
Stephen G. Gasiorowicz	1951
John H. Gay	1951
Robert K. Golden	1950
George E. Gourrich	1949
Harold Gruen	1949
Robert M. Hayes	1950
Urs W. Hochstrasser	1951
William C. Hoffman	1950
Kenneth E. Iverson	1951
Lloyd K. Jackson	1949
Thomas E. Kurtz	1951
Richard H. Lawson	1951
Eugene Levin	1953
Genevevo C. Lopez	1953
Howard W. Luchsinger	1948
Harold Luxenberg	1949
Stanley W. Mayer	1952
Edwin Mookini	1953
Michael J. Moracsik	1951
Mervin Muller	1953
Thomas Neill, Jr.	1951
Raymond P. Peterson, Jr.	1948
Lloyd Philipson	1953
Anthony Ralston	1951
Daniel B. Ray	1951
Theodore D. Schultz	1951
John Selfridge	1952
Robert H. Senhart	1948
Marvin L. Stein	1948
James G. C. Templeton	1949
Marion I. Walter	1952
William H. Warner	1951
Hans F. Weinberger	1949
James P. Wesley	1950
Mollie Z. Wirtschafer	1951
Roger D. Woods	1952

Computation and Mathematical Services Unit

The dates listed are the initial dates of employment.

Gertrude Blanch	1948
Louis L. Bailin	1948
Arnold D. Hestenes	1951
Frederick H. Hollander	1951
Marvin Howard	1949
Roselyn Siegel Lipkis	1948
Harold Luxenberg	1950
Leslie H. Miller	1951

William C. Randels	1950
Robert R. Reynolds	1950
Thomas H. Southard	1951
Everett C. Yowell	1948

Computing Staff

The dates listed are the initial dates of employment.

Helen Arens	1948
Warren A. Baily	1952
Jean H. Ballou	1952
Leonus T. Batiste	1951
Patricia Burton Bremer	1948
Patricia L. Childress	1952
Dennison A. Curtiss	1952
Michael Cohen	1950
Leola Cutler	1948
Richard L. Dunn	1952
Alexandra Forsythe	1950
Eileen M. Feyder	1948
Edward D. Fisher	1951
Lillian Forthal	1949
Gladys P. Franklin	1950
J. Russell Franks	1948
James W. Gardener	1952
Fannie M. Gordon	1948
Charles J. A. Halberg	1952
Benjamin F. Handy, Jr.	1952
Elizabeth Harding	1948
Robert M. Hayes	1950
Emma M. Henderson	1951
Donald L. Henley	1949
Delia Herbig	1948
Ruth B. Horgan	1948
William Justice, Jr.	1953
William P. Keating	1952
Gerald W. Kimble	1949
Mary M. Kruse	1951
Thomas D. Lakin	1950
Eugene Levin	1952
Barbara Levine	1952
Norma Logel	1950
Nancy Robbins Mann	1949
Shirley L. Marks	1948
Frank I. Meek	1953
Helen V. Meek	1953
Owen R. Mock	1951
Elmer E. Osborne	1951
William O. Paine, Jr.	1948
John A. Postley	1948
Mary M. Y. Quan	1952
Everett A. Rea	1948
Nan N. Reynolds	1951
La Verne E. Rickard	1951
Leon C. Robbins, Jr.	1948
Albert H. Rosenthal	1949
Philip H. Sayre	1952

William C. Schultz	1948
M. Winfred Smith	1949
Jerome A. Sneck	1951
Herbert Snow	1949
Seymour Soll	1951
Louise M. Straus	1951
Rose Tishman	1948
Mary Tudor	1949
Gladys P. Walker	1953
Maria Weber	1952
Marian E. Williams	1953
Lindley L. Wilson	1950
Boyd C. Zacharias	1948

Machine Development Unit

The dates listed are the initial dates of employment.

Harry D. Huskey	1948
Biago F. Ambrosio	1949
William Arsenault	1952
Edward I. Lacey	1949
Harry T. Larson	1949
Roselyn Siegel Lipkis	1948
Norman L. Loretz	1950
David S. Rutland	1949
Robert R. Schmidt	1953
Ragnar Thorensen	1951

Electronic Laboratory Staff

The dates listed are the initial dates of employment.

Brent H. Alford	1949
Robert V. Barnett	1949
Charles M. Brauer	1949
Arnold Dolmatz	1949
Blanche C. Eidem	1949
John P. Francis	1951
Allen E. Garfein	1952
Sidney S. Green	1949
Milton B. Grier	1949
Gaylord W. Jones	1951
Louis E. Justice	1951
Charles H. Keller	1952
John D. Mach	1949
Michael J. Markakis	1949
Howard K. Marks	1951
Edward D. Martinolich	1950
Kenneth J. Millikin	1951
Charles A. Mitchell II	1951
Lyle E. Mitchell	1951
Wallace J. Moore	1951
John L. Newberger	1949
Geraldine Orr	1949
Eugene M. Rector	1951
Dean W. Slaughter	1951
Sol Scope	1952
James W. Walsh	1949

Administrative Staff and Secretaries

The dates listed are the initial dates of employment. Initial classifications are given within parentheses except for secretaries. Some of the secretaries were given administrative posts later. Many served only for a short period.

Elsie A. Aho	1949	
James D. Allen	1952	(Clerk)
Louis H. Adelizzi	1952	(Office Machine Operator)
Wilbur W. Bolton	1948	(Administrative Officer)
Jean S. Booth	1950	
Gloria E. Bosowski	1950	
Eva Cassirer	1951	(Translator)
Harriet W. Cotton	1951	
Edna S. Cruzan	1948	
Mildred L. Dalton	1951	
Betty E. Dean	1951	
Selma S. Doumani	1950	
Mary G. Eskridge	1952	
Gloria E. Estes	1951	
Irene M. Everett	1949	
Albert H. Feldman	1951	(Office Machine Operator)
Ellma Franz	1948	(Library Assistant)
Mildred S. Goldberg	1948	
Sandra Goldberg	1952	
Vendla H. Gordanier	1951	
Margaret E. Gould	1949	
Eleanore J. Harris	1949	
Mark H. Hennes	1951	(Office Machine Operator)
Dorothy Hibbard	1949	
Miller A. Holt	1952	(Stock Clerk)
Velma R. Huskey	1949	(Physical Science Editor)
Elsie L. Husman	1948	
Dora Kaplan	1948	
Earle W. Kimball	1950	(Stock Clerk)
Sylvia C. Krasell	1948	
Cecelia R. Leonard	1950	
Dora K. Madoff	1950	(Administrative Assistant)
Mildred L. Martinolich	1952	
Tess D. Mazer	1949	
M. Evelyn Michaud	1948	(Administrative Assistant)
Lucy A. Moore	1951	
Irene M. Morgan	1949	
Leo Moskowitz	1952	(Procurement Clerk)
Anne B. Oates	1952	
Sata Ohashi	1949	
Patricia Kelly Peed	1948	(Librarian)
Gertrude Z. Reider	1948	
Elizabeth R. Schoenberg	1951	
Ruth A. Stafford	1950	
Estelle H. Strauss	1952	(Library Assistant)
Joyce R. Toy	1948	
Reve J. Vineyard	1950	
Mildred Vulkmanic	1952	(Administrative Assistant)
James R. Walker	1951	(Office Machine Operator)
Kathrine C. Warren	1950	
Mildred B. Webb	1952	(Administrative Clerk)
Madeline V. Youll	1950	(Library Assistant)

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11. ABSTRACT (A 200-WORD OR LESS FACTUAL SUMMARY OF MOST SIGNIFICANT INFORMATION. IF DOCUMENT INCLUDES A SIGNIFICANT BIBLIOGRAPHY OR LITERATURE SURVEY, MENTION IT HERE.)

This is a history of the Institute for Numerical Analysis (INA) with special emphasis on its research program during the period 1947 to 1956. The Institute for Numerical Analysis was located on the campus of the University of California, Los Angeles. It was a section of the National Applied Mathematics Laboratories, which formed the Applied Mathematics Division of the National Bureau of Standards (now the National Institute of Standards and Technology), under the U.S. Department of Commerce.

This history of the program at INA is concerned primarily with the development of mathematics pertinent to solving numerical computations. This development could happen only if some mathematicians were proficient in handling the electronic digital computers. To insure that there would be some, INA was constituted. It was well funded, and could attract first class mathematicians to take a year off for research at INA. There they were in the midst of people solving problems of considerable difficulty using the digital computers. They were thus enticed into using them. When this happened, many important developments emerged. This history is centered around these people and asks "Who were there?", "What were their interests?", "What did they do?".

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